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The analysis of void wave propagation in adiabatic monodispersed bubbly two-phase flows using an ensemble-averaged two-fluid model

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Abstract

A three-dimensional two-fluid model has been developed using ensemble-averaging techniques. The two-fluid model was closed for adiabatic two-phase bubbly flows using cell averaging which accounted for the dispersed phase distribution in the region of the averaging volume. The phasic interfacial momentum exchange includes the surface stress developed on the interface which is induced by the relative motion of the phases. The surface stress has been obtained by treating the interface as an elastic spherical shell. A characteristic analysis revealed that the one-dimensional system of two-fluid conservation equations which were derived is well-posed over a range of void fractions with increased value of the interfacial pressure. The propagation of void fraction disturbances (i.e. the void wave) has also been analyzed by performing a dispersion analysis. The speed, stability and damping of the linear void waves have been obtained. To study finite amplitude void waves, the system of equations has been transformed into a moving coordinate system, and asymptotic solutions of the transformed nonlinear void wave equation have been obtained. The speed and the stability of different types of nonlinear void waves have been found to be sensitive to the closure relations of the two-fluid model. Among the different constitutive parameters, the interfacial pressure difference in the continuous phase and the void fraction gradient in the non-drag force are found to be the most significant in determining behavior of void waves in bubbly flows. The derived void wave speed agrees well with the void wave data of bubbly air–water flow.

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1. Introduction

Different averaging techniques have been proposed for deriving two-fluid models. The time-average technique has been extensively studied by Ishii (1975) and has been widely used by many researchers. Nigmatulin (1979) derived a volumetric averaged set of balance equations which were constituted by using the concept of cell averaging. Delhaye (1976) developed a space/time averaging technique for one-dimensional two-phase flows, and Drew and Lahey (1989a) developed a three-dimensional two-fluid model using a combination of space and time averaging.

An ensemble-averaging technique is the most fundamentally rigorous form of averaging (Buyevich, 1971; Batchelor, 1970). In the ensemble average process, the ensemble is a set of flows that can occur at a specified position and time. Thus, the ensemble-averaging may include all the phasic interactions without specifying the time and length scales, in contrast to the other averaging techniques. Arnold (1988) developed a multidimensional two-fluid model using an ensemble-averaging technique.

In this work, a two-fluid model, which is an extension of the model derived by Arnold (1988), is derived. By considering the probability of a dispersed phase particle's location within an averaging volume (i.e. a cell) the two-fluid constitutive relations for adiabatic two-phase bubbly flows have been derived. The interfacial momentum transfers between the phases have been found by treating the phasic interface as an elastic spherical shell which experiences the force induced by the relative motion of the continuous phase.

To assess the validity of the two-fluid model presented herein, limiting steady and transient situations, in which the void fraction vanishes, have been considered. It is also shown that the two-fluid model which was derived is compatible with those previously derived two-fluid models by Geurst (1986), Wallis (1991), and Pauchon and Smereka (1992).

The well-posedness of the two-fluid model which has been derived is studied by considering the system's one-dimensional characteristics. The two-fluid model is found to be well-posed within a range of void fractions when the interfacial pressure is given to be greater than that of the ideal spherical bubble. If we take the value of the pressure coefficient, C_p, recommended by Lance and Bataille (1991), i.e. $C_p = 1.0$, the two-fluid model is well posed for a wide range of volume fraction.

Since the properties of the void wave have been found (Bouré, 1982; Pauchon and Banerjee, 1988; Park et al., 1990a; Biesheuvel and Gorrisen, 1990; Lahey, 1991) to be sensitive to the two-fluid model's closure relations, void wave propagation phenomenon has been analyzed.

Using the void wave dispersion model, the stability, speed and damping of the void wave have been determined. To analyze finite amplitude void waves, such as void wave shocks and solitons (Haley et al., 1991; Park et al., 1990b), the system of equations are cast into a moving coordinates. The speed and the stability of different nonlinear void waves has been found from the nonlinear void wave equations.

This study shows that an ensemble-averaged two-fluid model, which was constituted using a cell model approximation for dispersed two-phase flows, is appropriate for describing transient and steady phenomena in dilute dispersed two-phase flows.

2. Ensemble-averaged two-fluid equations

The ensemble-averaged two-fluid conservation equations for adiabatic two-phase flows can be obtained by averaging the phasic local instantaneous mass and momentum conservation equations (Drew, 1983; Lahey and Drew, 1990) as:

Mass conservation:

$$\frac{\partial}{\partial t}(\epsilon_k \rho_k) + \nabla \cdot (\epsilon_k \rho_k \mathbf{v}_k) = 0. \tag{1}$$

Momentum conservation:

$$\frac{\partial}{\partial t}(\epsilon_k \rho_k \mathbf{v}_k) + \nabla \cdot (\epsilon_k \rho_k \mathbf{v}_k \mathbf{v}_k) = -\nabla(\epsilon_k p_k) + \nabla \cdot [\epsilon_k(\tau_k + \tau_k^{Re})] + \epsilon_k \rho_k \mathbf{g} + \mathbf{M}_{ki} \tag{2}$$

where ρ_k, p_k and τ_k are averaged variables weighted with the phase indicator function, e.g.

$$\rho_k \triangleq \overline{\chi_k \tilde{\rho}_k}/\epsilon_k, \tag{3}$$

where $\tilde{\rho}_k$ is the exact density. Also, $\mathbf{v}_k$ is the mass-weighted average velocity, defined by

$$\mathbf{v}_k \triangleq \overline{\chi_k \tilde{\rho}_k \tilde{\mathbf{v}}_k}/\epsilon_k \rho_k \tag{4}$$

where, $\tilde{\mathbf{v}}_k$ is the exact velocity field. Also,

$$\tau_k^{Re} \triangleq \overline{-\chi_k \tilde{\rho}_k \mathbf{v}_k' \mathbf{v}_k'} \tag{5}$$

$$\mathbf{M}_{ki} \triangleq -\overline{\mathbf{T}_k \cdot \nabla \chi_k} \tag{6}$$

and the phase indicator function has been defined as

$$\chi_k(\mathbf{x}, t) \triangleq \begin{cases} 1, & \text{if phase-}k\text{ is found at } (\mathbf{x}, t) \\ 0, & \text{otherwise} \end{cases}.$$

We note that the gradient of the phase indicator function can be expressed as:

$$\nabla \chi_k(\mathbf{x}, t) = -\mathbf{n}_k \delta_k(\mathbf{x}, t) \tag{7}$$

where $\mathbf{n}_k$ is the unit normal vector and $\delta_k(\mathbf{x}, t)$ is a Dirac delta function.

If we treat the phasic interface as an elastic shell of infinitesimal thickness which experiences the stress induced on it by each phase, we obtain the momentum equation for the shell (i.e. the interface) as:

$$\nabla \cdot \mathbf{T}_s = 0, \tag{8}$$

where

$$\mathbf{T}_s \triangleq \mu_s[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + \lambda_s(\nabla \cdot \mathbf{u})\mathbf{I} \tag{9}$$

and, $\mathbf{u}$, μ_s and λ_s are the displacement and the Lamé constants, respectively.

To obtain the so-called *momentum jump condition* between the phases, we must appropriately average the interfacial momentum equation, Eq. (8). If we define the indicator function for the interface (i.e. the shell) as:

$$\chi_s(\mathbf{x}, t) \triangleq \begin{cases} 1, & \text{if the shell is found at } (\mathbf{x}, t) \\ 0, & \text{otherwise} \end{cases}$$

we obtain the averaged form of Eq. (8) as

$$\overline{\nabla \cdot (\chi_s \mathbf{T}_s)} = \overline{\mathbf{T}_s \cdot \nabla \chi_s}. \tag{10}$$

The shell indicator function can be related to the phasic indicator functions for the gas (G) and the liquid (L) phases as:

$$\chi_s = 1 - \chi_G - \chi_L \tag{11}$$

Using Eq. (11), we can rearrange Eq. (10) as

$$\overline{\nabla \cdot (\chi_s \mathbf{T}_s)} = -\overline{\mathbf{T}_s \cdot \nabla \chi_G} - \overline{\mathbf{T}_s \cdot \nabla \chi_L} \tag{12}$$

or, using Eq. (7).

$$\overline{\nabla \cdot (\chi_s \mathbf{T}_s)} = \overline{\mathbf{T}_s \cdot \mathbf{n}_G \delta_G(\mathbf{x}, t)} + \overline{\mathbf{T}_s \cdot \mathbf{n}_L \delta_L(\mathbf{x}, t)}. \tag{13}$$

We now may introduce an assumption that the normal component of the stress at the surface of the shell (i.e. the interface) is continuous, that is:

$$(\mathbf{T}_s)_{in} \cdot \mathbf{n}_G = \mathbf{T}_{Gi} \cdot \mathbf{n}_G \tag{14}$$

at the inner surface, and

$$(\mathbf{T}_s)_{ex} \cdot \mathbf{n}_L = \mathbf{T}_{Li} \cdot \mathbf{n}_L \tag{15}$$

at the outer surface. Here, the subscripts ‘in’ and ‘ex’ indicate the interior and the exterior surfaces of the shell, respectively.

Inserting Eqs. (14) and (15) into Eq. (13), we obtain

$$\overline{\nabla \cdot (\chi_s \mathbf{T}_s)} = \overline{\mathbf{T}_G \cdot \mathbf{n}_G \delta_G(\mathbf{x}, t)} + \overline{\mathbf{T}_L \cdot \mathbf{n}_L \delta_L(\mathbf{x}, t)} \tag{16}$$

or, equivalently,

$$\overline{\nabla \cdot (\chi_s \mathbf{T}_s)} = -\overline{\mathbf{T}_G \cdot \nabla \chi_G} - \overline{\mathbf{T}_L \cdot \nabla \chi_L}. \tag{17}$$

If we use the definitions given by Eq. (6), we obtain

$$\mathbf{M}_{Gi} + \mathbf{M}_{Li} = \overline{\nabla \cdot (\chi_s \mathbf{T}_s)} = \nabla \cdot \overline{(\chi_s \mathbf{T}_s)}, \tag{18}$$

which can be considered to be a *momentum jump condition* between the phases.

As can be seen in Eqs. (3)–(6) and (18), the terms which arise from the ensemble averaging process do not explicitly involve the basic two-fluid state variables (e.g. p_k, $\mathbf{v}_k$ and ϵ_k). Thus, these equations must be constituted in order to achieve closure. Moreover, it is important to

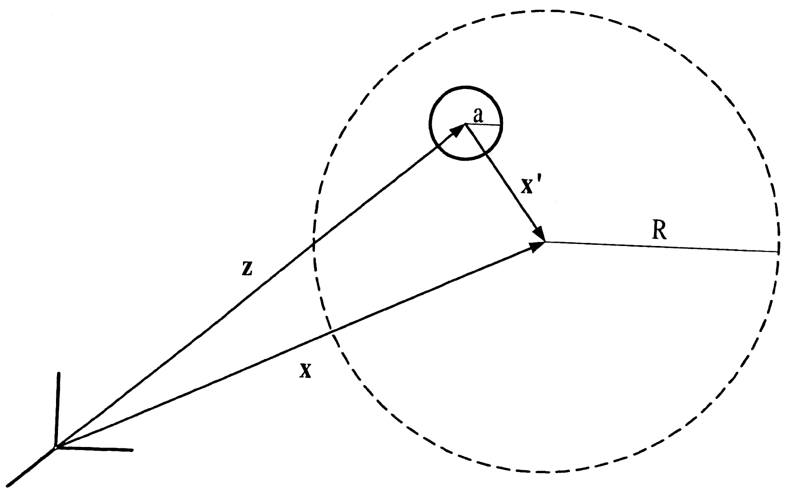

Fig. 1. A typical averaging cell.

realize that it is with these closure laws that the microscale information which was lost during the averaging procedure is reintroduced (Alajbegovic, 1994).

3. Constitutive relations for bubbly flows

The ensemble averaged two-fluid equations can be constituted by using the cell averaging technique of Arnold (1988) with the following assumptions:

(i) The fluids are inviscid, incompressible and have constant thermophysical properties;
(ii) The dispersed phase can be treated as a dilute dispersion of spheres.
(iii) The nonuniformity in the distribution of the dispersed phase is small.

In the cell model ensemble average, the ensemble is that set of flows that can occur at location **x** with the center of the spherical bubble occupying different positions within the cell (see Fig. 1). In this case, the average is performed by integrating over the variable, **z**, that is, the possible positions that the center of the bubble can have. The center of the bubble, **z**, can lie anywhere inside the sphere of radius R. We assume that the distribution of positions is such that,

$$P(\mathbf{x}, t; \mathbf{z}) = \frac{\mathrm{d}V}{\frac{4}{3}\pi R^3}\left(1 - \mathbf{x}' \cdot \frac{\nabla \epsilon_G(\mathbf{x}, t)}{\epsilon_G(\mathbf{x}, t)}\right) \tag{19}$$

is the probability of finding a bubble in a volume $\mathrm{d}V$ surrounding the point **x** where $\mathbf{x}' = \mathbf{x} - \mathbf{z}$. This is approximately equal to

$$\frac{\mathrm{d}V}{\frac{4}{3}\pi R^3}\frac{\epsilon_G(\mathbf{z}, t)}{\epsilon_G(\mathbf{x}, t)}.$$

The factor $\epsilon_G(\mathbf{z}, t)/\epsilon_G(\mathbf{x}, t)$ is the appropriate multiplier to change the probability of finding the bubble at $\mathbf{x}$ to the probability of finding the bubble at $\mathbf{x}$ to the probability of finding the bubble at $\mathbf{z}$.

Note that the 'cell model' accounts to the influence of one bubble on the quantities calculated at a point $\mathbf{x}$. Thus, the method ignores effects of order ϵ_G^2. In addition, we have systematically ignored terms resulting from products of gradients. These terms are of order $(a/R)^2$ smaller than terms retained.

For inviscid, irrotational flows, the flow potential ϕ satisfies

$$\nabla^2\phi = 0 \tag{20}$$

and the corresponding pressure in the continuous phase is given by the transient Bernoulli equation:

$$\mathrm{P_L} = p_\infty - \rho_\mathrm{L}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2\right). \tag{21}$$

If we consider a spherical bubble of radius 'a' moving with velocity $\mathbf{v}_G$ $(\mathbf{z}, t)$ in a flow field, $\mathbf{v}_{Lo}(t) + \mathbf{x}' \cdot \nabla\mathbf{v}_L$, the appropriate velocity potential is (Voinov, 1973):

$$\phi(\mathbf{x};\mathbf{z}) = -\frac{a^2}{r'}\dot{a} - \frac{1}{2}\left(\frac{a}{r'}\right)^3 \mathbf{v}_G\cdot\mathbf{x}' + \phi_o + \left[1 + \frac{1}{2}\left(\frac{a}{r'}\right)^3\right]\nabla\phi_o\cdot\mathbf{x}'$$

$$+\left[\frac{1}{2} + \frac{2}{3}\left(\frac{a}{r'}\right)^5\right]\nabla\nabla\phi_o : \mathbf{x}'\mathbf{x}'\ldots \tag{22}$$

where $r' = |\mathbf{x}'|$ and ϕ_o is the flow potential at $\mathbf{z}$ when the bubble does not exist.

If we assume that the average flow around the bubble can be approximated by a uniform velocity gradient, we may expand ϕ_o as,

$$\phi_o = \phi' - \nabla\phi'\cdot\mathbf{x}' + \ \ldots \tag{23}$$

where ϕ' is the flow potential at $\mathbf{x}$ for the undisturbed flow.

Using Eq. (23), we may rewrite Eq. (22) as:

$$\phi(\mathbf{x};\mathbf{z}) = \phi' - \frac{a^2}{r'}\dot{a} - \frac{1}{2}\left(\frac{a}{r'}\right)^3 \mathbf{v}_r\cdot\mathbf{x}' + \frac{1}{2}\left[-\left(\frac{a}{r'}\right)^3 + \frac{2}{3}\left(\frac{a}{r'}\right)^5\right]\nabla\nabla\phi : \mathbf{x}'\mathbf{x}' + \ldots \tag{24}$$

where,

$$\mathbf{v}_r \triangleq \mathbf{v}_G - \nabla\phi' \tag{25}$$

is the relative velocity of the bubble at the sphere center.

The local velocity of the continuous phase can be obtained from Eq. (24) as,

$$\tilde{\mathbf{v}}_L(\mathbf{x};\mathbf{z}) = \nabla\phi(\mathbf{x};\mathbf{z})$$

$$= \mathbf{V} + \frac{1}{2}\left(\frac{a}{r'}\right)^3\left[\frac{3(\mathbf{v}_r\cdot\mathbf{x}')\mathbf{x}'}{(r')^2} - \mathbf{v}_r\right] + \frac{1}{3}\left(\frac{a}{r'}\right)^5\mathbf{x}'\cdot\left[\frac{5(\nabla\mathbf{v}_L\cdot\mathbf{x}'\mathbf{x}')}{(r')^2} - 2\mathbf{v}_L\right] \tag{26}$$

where,

$$\mathbf{V} = \nabla\phi'. \tag{27}$$

Note that the liquid velocity, in the absence of the bubble, can depend on $\mathbf{x}$. Moreover, the bubble velocity $\mathbf{v}_b$ can depend on position [in this case $\mathbf{v}_b = \mathbf{v}_b(\mathbf{z})$], and the volume fraction of the gas phase can also depend on position. The cell model provides a means of accounting for all these effects.

To average the continuous phase velocity, we place a large sphere (i.e. the cell) of radius, R centered at $\mathbf{x}$. We then obtain the cell-average velocity of the continuous phase as:

$$\mathbf{v}_L = \frac{1}{\frac{4}{3}\pi(R^3 - a^3)}\int_a^R\iint_{\Omega(r')}\tilde{\mathbf{v}}_L(\mathbf{x};\mathbf{z})\left(1 - \frac{\mathbf{x}'\cdot\nabla\epsilon_G}{\epsilon_G}\right)\mathrm{d}\Omega(r')\,\mathrm{d}r'. \tag{28}$$

Evaluating the integral using Eq. (26), we obtain

$$\mathbf{v}_L = \mathbf{V}, \tag{29}$$

where the products of the derivatives may be neglected according to the assumption (ii) made in this section.

The Reynolds stress for the continuous phase is defined by Eq. (4) as

$$\begin{aligned}\epsilon_L\tau_L^{Re} &= -\overline{\chi_L\rho_L\mathbf{v}_L'\mathbf{v}_L'}\\ &= \frac{1}{\frac{4}{3}\pi(R^3 - a^3)}\int_a^R\iint_{\Omega(r')} -\rho_L\mathbf{v}_L'\mathbf{v}_L'\left(1 - \frac{\mathbf{x}'\cdot\nabla\epsilon_G}{\epsilon_G}\right)\mathrm{d}\Omega(r')\,\mathrm{d}r',\end{aligned} \tag{30}$$

where

$$\mathbf{v}_L' = \nabla\phi(\mathbf{x};\mathbf{z}) - \mathbf{v}_L. \tag{31}$$

Using Eqs. (24) and (31), we obtain

$$\tau_L^{Re} = -\frac{1}{20}\rho_L\frac{\epsilon_G}{\epsilon_L}[\mathbf{v}_r\mathbf{v}_r + 3(\mathbf{v}_r\cdot\mathbf{v}_r)\mathbf{I}] \tag{32}$$

which, for sufficiently small ϵ_G, reduces to the results previously obtained by Arnold (1988) and Biesheuvel and Wijngaarden (1984).

Similarly, the interfacial averaged Reynolds stress is,

$$\epsilon_L \tau_{Li}^{Re} = -\chi_L \overline{\rho_L \mathbf{v}'_{Li} \mathbf{v}'_{Li}} = \frac{1}{4\pi a^2} \int\int_{\Omega(a)} -\rho_L \mathbf{v}'_{Li} \mathbf{v}'_{Li} \mathrm{d}\Omega(a), \tag{33}$$

where $\mathbf{v}'_{Li}$ is the deviation of the continuous phase velocity at the surface of the sphere (i.e. at $r' = a$). (Note that we do not include the variable part of the probability of finding the sphere at $\mathbf{z}$—it is negligible.)

Evaluating the integral in Eq. (33), we obtain:

$$\tau_{Li}^{Re} = -\frac{1}{20} \rho_L [\mathbf{v}_r \mathbf{v}_r + 3(\mathbf{v}_r \cdot \mathbf{v}_r)\mathbf{I}]. \tag{34}$$

If we introduce Eq. (24) into Eq. (21), we obtain the local pressure in the continuous phase as,

$$\begin{aligned}\tilde{p}(\mathbf{x}; \mathbf{z}) = p_o &+ \frac{1}{2}\left(\frac{a}{r}\right)^3 \rho_L \dot{\mathbf{v}}_r \cdot \mathbf{x}' - \left[\left(\frac{a}{r'}\right)^5 - 2\left(\frac{a}{r'}\right)^2\right] \rho_L \dot{a} \frac{\mathbf{v}_r \cdot \mathbf{x}'}{r'} \\ &- \left[\frac{1}{8}\left(\frac{a}{r'}\right)^6 + \frac{1}{2}\left(\frac{a}{r'}\right)^3\right] \rho_L \mathbf{v}_r \cdot \mathbf{v}_r + \left[\frac{3}{2}\left(\frac{a}{r'}\right)^3 - \frac{3}{8}\left(\frac{a}{r'}\right)^6\right] \rho_L \frac{(\mathbf{v}_r \cdot \mathbf{x}')^2}{(r')^2} \\ &+ \left[\frac{3}{2}\left(\frac{a}{r'}\right)^3 + \frac{5}{3}\left(\frac{a}{r'}\right)^5 - \frac{2}{3}\left(\frac{a}{r'}\right)^7\right] \rho_L \frac{(\nabla \mathbf{V} : \mathbf{x}'\mathbf{x}')(\mathbf{v}_r \cdot \mathbf{x}')}{(r')^2} \\ &+ \left[\frac{1}{2}\left(\frac{a}{r'}\right)^3 + \frac{2}{3}\left(\frac{a}{r'}\right)^5 + \frac{1}{3}\left(\frac{a}{r'}\right)^8\right] \rho_L (\nabla \mathbf{V} : \mathbf{x}'\mathbf{v}_r) \\ &+ \frac{1}{2}\left(\frac{a}{r'}\right)^3 \rho_L (\nabla \mathbf{V} : \mathbf{x}'\mathbf{v}_r) \\ &+ \left[\frac{3}{4}\left(\frac{a}{r'}\right)^6 - 3\left(\frac{a}{r'}\right)^3\right] \rho_L \frac{(\nabla \mathbf{v}_r : \mathbf{x}'\mathbf{x}')(\mathbf{v}_r \cdot \mathbf{x}')}{(r')^2} \\ &+ \left[\left(\frac{a}{r'}\right)^3 + \frac{1}{4}\left(\frac{a}{r'}\right)^6\right] \rho_L (\nabla \mathbf{v}_r : \mathbf{x}'\mathbf{v}_r)\end{aligned} \tag{35}$$

where, the continuous phase pressure if the bubble were not present is:

$$p_o \triangleq p_\infty - \rho \dot{\phi}' - \frac{1}{2} \rho_L \mathbf{V} \cdot \mathbf{V}. \tag{36}$$

Thus, the average pressure of the continuous phase is given by:

$$p_{\mathrm{L}} = \frac{1}{\frac{4}{3}\pi(R^3 - a^3)} \int_a^R \iint_{\Omega(a)} \tilde{p}(\mathbf{x};\mathbf{z}) \left(1 - \frac{\mathbf{x}' \cdot \nabla\epsilon_{\mathrm{G}}}{\epsilon_{\mathrm{G}}}\right) \mathrm{d}\Omega(r')\,\mathrm{d}r'. \tag{37}$$

Evaluating the integral in Eq. (37), using Eq. (35), we obtain (Arnold, 1988),

$$p_{\mathrm{L}} = p_{\mathrm{o}} - \frac{1}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}} \cdot \mathbf{v}_{\mathrm{r}}. \tag{38}$$

The continuous phase interfacial pressure (i.e. the pressure at the exterior surface of the spherical shell) can be found by setting $r' = a$ in Eq. (35), resulting in,

$$\begin{aligned}\tilde{p}_{\mathrm{Li}} = {} & p_{\mathrm{o}} + \frac{1}{2}\rho_{\mathrm{L}} a\mathbf{v}_{\mathrm{r}} \cdot \mathbf{e}_{\mathrm{r}} + \frac{3}{2}\rho_{\mathrm{L}}\dot{a}\mathbf{v}_{\mathrm{r}} \cdot \mathbf{e}_{\mathrm{r}} - \frac{5}{8}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}} \cdot \mathbf{v}_{\mathrm{r}} + \frac{9}{8}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}} \cdot \mathbf{e}_{\mathrm{r}})^2 \\ & + \frac{5}{2}\rho_{\mathrm{L}} a(\nabla\mathbf{V} : \mathbf{e}_{\mathrm{r}}\mathbf{e}_{\mathrm{r}})(\mathbf{v}_{\mathrm{r}} \cdot \mathbf{e}_{\mathrm{r}}) + \frac{3}{2}\rho_{\mathrm{L}} a(\nabla\mathbf{V} : \mathbf{e}_{\mathrm{r}}\mathbf{v}_{\mathrm{r}}) \\ & + \frac{1}{2}\rho_{\mathrm{L}} a(\nabla\mathbf{v}_{\mathrm{r}} : \mathbf{e}_{\mathrm{r}}\mathbf{v}_{\mathrm{r}}) - \frac{9}{4}\rho_{\mathrm{L}}(\nabla\mathbf{v}_{\mathrm{r}} : \mathbf{e}_{\mathrm{r}}\mathbf{e}_{\mathrm{r}})(\mathbf{v}_{\mathrm{r}} \cdot \mathbf{e}_{\mathrm{r}}) + \frac{5}{4}\rho_{\mathrm{L}} a(\nabla\mathbf{v}_{\mathrm{r}} : \mathbf{e}_{\mathrm{r}}\mathbf{v}_{\mathrm{r}}),\end{aligned} \tag{39}$$

where

$$\mathbf{e}_{\mathrm{r}} \triangleq \mathbf{x}'/r'.$$

The average interfacial pressure can be defined as,

$$p_{\mathrm{Li}} = \frac{1}{4\pi a^2} \iint_{\Omega(a)} \tilde{p}_{\mathrm{Li}} \mathrm{d}\Omega(a). \tag{40}$$

Performing the integration in Eq. (40), we obtain,

$$p_{\mathrm{Li}} = p_{\mathrm{o}} - \frac{1}{4}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}} \cdot \mathbf{v}_{\mathrm{r}}. \tag{41}$$

Thus, we obtain the interfacial pressure difference of the continuous phase, from Eqs. (38) and (41), as

$$\Delta p_{\mathrm{Li}} \triangleq p_{\mathrm{Li}} - p_{\mathrm{L}} = -\frac{1}{4}\rho_{\mathrm{L}}\epsilon_{\mathrm{L}}\mathbf{v}_{\mathrm{r}} \cdot \mathbf{v}_{\mathrm{r}} = -\frac{1}{4}\rho_{\mathrm{L}}(1 - \epsilon_{\mathrm{G}})\mathbf{v}_{\mathrm{r}} \cdot \mathbf{v}_{\mathrm{r}}, \tag{42}$$

which agrees with the result obtained by Stuhmiller (1977) when ϵ_{G} is sufficiently small.

One may also allow a deviation in the local velocity of the bubble, $\mathbf{v}_{\mathrm{b}}$, from the average velocity of the dispersed phase. However, it has been found (Drew, 1991) that these effects are negligible when the dispersed phase has a much lower density than the continuous phase (i.e. gas bubbles in liquid). Thus, in this work we have assumed:

$$\mathbf{v}_{\mathrm{G}} = \mathbf{v}_{\mathrm{b}}. \tag{43}$$

The interfacial momentum source for the liquid phase can be written as

$$\mathbf{M}_{\mathrm{Li}} = \overline{\tilde{p}_{\mathrm{L}}\nabla\chi_1} - \overline{\tilde{\tau}_{\mathrm{L}}\cdot\nabla\chi_{\mathrm{L}}}. \tag{44}$$

The first term in Eq. (44) can be evaluated as

$$\overline{\tilde{p}_{\mathrm{L}}\nabla\chi_{\mathrm{L}}} = \frac{1}{\frac{4}{3}\pi R^3}\int\int_{\Omega(a)} \tilde{p}_{\mathrm{Li}}\mathbf{e}_{\mathrm{r}}\left(1 - a\frac{\mathbf{e}_{\mathrm{r}}\cdot\nabla\epsilon_{\mathrm{G}}}{\epsilon_{\mathrm{G}}}\right)\mathrm{d}\Omega(a). \tag{45}$$

Using the continuous phase interfacial pressure given by Eq. (39), we obtain:

$$\begin{aligned}\overline{\tilde{p}_{\mathrm{L}}\nabla\chi_{\mathrm{L}}} =& \frac{1}{2}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{a}_{\mathrm{vm}} + \frac{1}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot(\nabla\mathbf{v}_{\mathrm{G}}^{\mathrm{T}} - \nabla\mathbf{v}_{\mathrm{G}})\\ &+ \frac{3}{2}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\frac{\dot{a}}{a}\mathbf{v}_{\mathrm{r}} + \frac{5}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot\nabla\mathbf{v}_{\mathrm{L}}^{\mathrm{T}}\\ &- \frac{9}{20}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}[\mathbf{v}_{\mathrm{r}}\cdot(\nabla\mathbf{v}_{\mathrm{G}} + \nabla\mathbf{v}_{\mathrm{G}}^{\mathrm{T}}) + (\nabla\cdot\mathbf{v}_{\mathrm{G}})\mathbf{v}_{\mathrm{r}}]\\ &- p_{\mathrm{o}}\nabla\epsilon_{\mathrm{G}} + \frac{2}{5}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}}\cdot\mathbf{v}_{\mathrm{r}})\nabla\epsilon_{\mathrm{G}} - \frac{9}{20}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}}\cdot\nabla\epsilon_{\mathrm{G}})\mathbf{v}_{\mathrm{r}},\end{aligned} \tag{46}$$

where

$$\mathbf{a}_{\mathrm{vm}} = \left(\frac{\partial}{\partial t} + \mathbf{v}_{\mathrm{b}}\cdot\nabla\right)\mathbf{v}_{\mathrm{b}} - \left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} \tag{47}$$

and we can interpret the time derivative of the bubble radius as,

$$\dot{a} = \left(\frac{\partial}{\partial t} + \mathbf{v}_{\mathrm{G}}\cdot\nabla\right)a \triangleq \frac{D_{\mathrm{G}}a}{Dt}. \tag{48}$$

Using the dispersed phase continuity equation, we obtain,

$$\frac{3}{2}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\epsilon_{\mathrm{G}}\frac{\dot{a}}{a} = \frac{1}{2}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\frac{D_{\mathrm{G}}\epsilon_{\mathrm{G}}}{Dt} + \frac{1}{2}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\epsilon_{\mathrm{G}}(\nabla\cdot\mathbf{v}_{\mathrm{G}}) \tag{49}$$

Using Eq. (49), we can rewrite Eq. (46) as:

$$\begin{aligned}\overline{\tilde{p}_{\mathrm{L}}\nabla\chi_{\mathrm{L}}} =& \frac{1}{2}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{a}_{\mathrm{vm}} + \frac{1}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot(\nabla\mathbf{v}_{\mathrm{G}}^{\mathrm{T}} - \nabla\mathbf{v}_{\mathrm{G}})\\ &+ \frac{1}{2}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\left(\frac{D_{\mathrm{G}}\epsilon_{\mathrm{G}}}{Dt} + \epsilon_{\mathrm{G}}\nabla\cdot\mathbf{v}_{\mathrm{G}}\right) + \frac{5}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot\nabla\mathbf{v}_{\mathrm{r}}^{\mathrm{T}}\\ &- \frac{9}{20}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}[\mathbf{v}_{\mathrm{r}}\cdot(\nabla\mathbf{v}_{\mathrm{r}} + \nabla\mathbf{v}_{\mathrm{r}}^{\mathrm{T}}) + (\nabla\cdot\mathbf{v}_{\mathrm{r}})\mathbf{v}_{\mathrm{r}}]\\ &- \left(p_{\mathrm{Li}} + \frac{1}{4}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot\mathbf{v}_{\mathrm{r}}\right)\nabla\epsilon_{\mathrm{G}} + \frac{2}{5}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}}\cdot\mathbf{v}_{\mathrm{r}})\nabla\epsilon_{\mathrm{G}}\\ &- \frac{9}{20}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}}\cdot\nabla\epsilon_{\mathrm{L}})\mathbf{v}_{\mathrm{r}}.\end{aligned} \tag{50}$$

The second term in Eq. (44) quantifies the viscous stress at the exterior surface of the spherical shell. Thus, one may try to find the local stress field in the liquid, τ_L, by analyzing the boundary layer around the spherical shell (Arnold, 1988). However, the viscous stress is only important when the relative velocity between the phases is small. Instead, the average stress induced by the motion of the continuous phase, which occurs when the boundary layer around the bubble separates, is dominant in the range of normal applications of two-phase bubbly flows. However, this effect is not considered in the inviscid analysis leading to Eq. (50). Therefore, it appears to be reasonable to introduce an interfacial drag model which includes both of the local viscous shear and the form drag around the spherical shell.

We may partition the interfacial momentum source for the continuous phase as (Lahey and Drew, 1990; Ishii and Mishima, 1984):

$$\mathbf{M}_{\mathrm{Li}} \triangleq \mathbf{M}_{\mathrm{Li}}^{(\mathrm{nd})} + \mathbf{M}_{\mathrm{Li}}^{(\mathrm{d})} + p_{\mathrm{Li}}\nabla\epsilon_{\mathrm{L}} - (\tau_{\mathrm{Li}} + \tau_{\mathrm{Li}}^{Re})\cdot\nabla\epsilon_{\mathrm{L}} \tag{51}$$

where,

$$\mathbf{M}_{\mathrm{Li}}^{(\mathrm{nd})} \triangleq \overline{(\tilde{p}_{\mathrm{L}} - p_{\mathrm{Li}})^{(\mathrm{nd})}\nabla\chi_{\mathrm{L}}} - \overline{(\tilde{\tau}_{\mathrm{L}} - \tau_{\mathrm{Li}}^{Re})\cdot\nabla\chi_{\mathrm{L}}} \tag{52}$$

or,

$$\begin{aligned}\mathbf{M}_{\mathrm{Li}}^{(\mathrm{nd})} &= \frac{1}{2}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{a}_{\mathrm{vm}} + \frac{1}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot(\nabla\mathbf{v}_{\mathrm{G}}^{\mathrm{T}} - \nabla\mathbf{v}_{\mathrm{G}}) \\ &+ \frac{1}{2}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\left(\frac{D_{\mathrm{G}}\epsilon_{\mathrm{G}}}{Dt} + \epsilon_{\mathrm{G}}\nabla\cdot\mathbf{v}_{G}\right) + \frac{5}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot\nabla\mathbf{v}_{\mathrm{r}}^{\mathrm{T}} \\ &- \frac{9}{20}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}[\mathbf{v}_{\mathrm{r}}\cdot(\nabla\mathbf{v}_{\mathrm{r}} + \nabla\mathbf{v}_{\mathrm{r}}^{\mathrm{T}}) + (\nabla\cdot\mathbf{v}_{\mathrm{r}})\mathbf{v}_{\mathrm{r}}] \\ &+ \frac{3}{10}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}}\cdot\mathbf{v}_{\mathrm{r}})\nabla\epsilon_{\mathrm{G}} - \frac{2}{5}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}}\cdot\nabla\epsilon_{\mathrm{G}})\mathbf{v}_{\mathrm{r}}\end{aligned} \tag{53}$$

$$\mathbf{M}_{\mathrm{Li}}^{(\mathrm{d})} \triangleq -\overline{(\tilde{\tau}_{\mathrm{L}} - \tau_{\mathrm{Li}})\cdot\nabla\chi_{\mathrm{L}}} + \overline{(\tilde{p}_{\mathrm{L}} - p_{\mathrm{Li}})^{(\mathrm{d})}\nabla\chi_{\mathrm{L}}} \tag{54}$$

It is also well known (Drew and Lahey, 1987, 1989b) that a lateral force induced by the rotational part of the liquid flow field around the sphere (i.e. the so-called lift force) should be included in the non-drag momentum exchange term.

The lift force was found by Drew and Lahey (1987, 1989b) to be:

$$\mathbf{M}_{\mathrm{Li}}^{\mathrm{L}} = C_{\mathrm{L}}\rho_{\mathrm{L}}\epsilon_{\mathrm{G}}\mathbf{v}_{\mathrm{r}}\times\nabla\times\mathbf{v}_{\mathrm{L}} \tag{55}$$

where, depending on the flow conditions, $C_{\mathrm{L}} = 0.01 - 0.5$.

Thus, if we include the lift force in Eq. (53), we obtain,

$$\mathbf{M}_{\mathrm{Li}}^{(\mathrm{nd})} = \frac{1}{2}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{a}_{\mathrm{vm}} + \frac{1}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot(\nabla\mathbf{v}_{\mathrm{G}}^{\mathrm{T}} - \nabla\mathbf{v}_{\mathrm{G}})$$

$$+\frac{1}{2}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\left(\frac{D_{\mathrm{G}}\epsilon_{\mathrm{G}}}{Dt} + \epsilon_{\mathrm{G}}\nabla\cdot\mathbf{v}_{\mathrm{G}}\right) + \frac{5}{4}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{r}}\cdot\nabla\mathbf{v}_{\mathrm{r}}^{\mathrm{T}}$$

$$-\frac{9}{20}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}[\mathbf{v}_{\mathrm{r}}\cdot(\nabla\mathbf{v}_{\mathrm{r}} + \nabla\mathbf{v}_{\mathrm{r}}^{\mathrm{T}}) + (\nabla\cdot\mathbf{v}_{\mathrm{r}})\mathbf{v}_{\mathrm{r}}]$$

$$+\frac{3}{10}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}}\cdot\mathbf{v}_{\mathrm{r}})\nabla\epsilon_{\mathrm{G}} - \frac{2}{5}\rho_{\mathrm{L}}(\mathbf{v}_{\mathrm{r}}\cdot\nabla\epsilon_{\mathrm{G}})\mathbf{v}_{\mathrm{r}}$$

$$+C_{\mathrm{L}}\rho_{\mathrm{L}}\epsilon_{\mathrm{G}}\mathbf{v}_{\mathrm{r}}\times\nabla\times\mathbf{v}_{\mathrm{L}}. \quad (56)$$

For monodispersed bubbly flow, the sum of the viscous shear force and the form drag can be modeled as,

$$\mathbf{M}_{\mathrm{Li}}^{(\mathrm{d})} \triangleq -\overline{(\tilde{\tau}_{\mathrm{L}} - \tau_{\mathrm{Li}})\cdot\nabla\chi_{\mathrm{L}}} + \overline{(\tilde{p}_{\mathrm{L}} - p_{\mathrm{Li}})^{(\mathrm{d})}\nabla\chi_{\mathrm{L}}} = \frac{3}{8}\frac{C_{\mathrm{D}}}{R_{\mathrm{b}}}\rho_{\mathrm{L}}\epsilon_{\mathrm{G}} \mid \mathbf{v}_{\mathrm{r}} \mid \mathbf{v}_{\mathrm{r}} \quad (57)$$

where the subscript (d) implies drag. The parameter C_{D} is an appropriate interfacial drag coefficient and R_{b} is the radius of the bubbles. It should be noted that we may also write, $C_{\mathrm{D}} = 8/3\ R_{\mathrm{b}}/D_{\mathrm{H}}\, f_i$, where f_i is the so-called interfacial friction factor.

Harmathy (1961) proposed a model for the drag coefficient for distorted bubbles as,

$$C_{\mathrm{D}} = \frac{4}{3}R_{\mathrm{b}}\left[\frac{G(\rho_{\mathrm{L}} - \rho_{\mathrm{G}})}{\sigma(1-\epsilon_{\mathrm{G}})}\right]^{1/2}. \quad (58)$$

For undistorted spherical bubbles, the interfacial drag coefficient proposed by Ishii and Zuber (1979) is:

$$C_{\mathrm{D}} = 24\frac{(1 + 0.1Re_{\mathrm{b}}^{0.75})}{Re_{\mathrm{b}}}, \quad (59)$$

where

$$Re_{\mathrm{b}} = \frac{2\rho_{\mathrm{L}} \mid \mathbf{v}_{\mathrm{r}} \mid R_{\mathrm{b}}}{\mu_{\mathrm{L}}}\epsilon_{\mathrm{L}}^{2.5\mu_{\mathrm{m}}^{*}} \quad (60)$$

$$\mu_{\mathrm{m}}^{*} = \frac{(\mu_{\mathrm{G}} + 0.4\mu_{\mathrm{L}})}{(\mu_{\mathrm{G}} + \mu_{\mathrm{L}})}. \quad (61)$$

Arnold (1988) constituted τ_{Li}, for different values of Reynolds number to obtain,

$$\tau_{\mathrm{Li}} = \mathbf{0}. \quad (62)$$

However, as shown in Eq. (34), this does not imply that the interfacial Reynolds stress, τ_{Li}^{Re}, is zero.

The average stress on the interface (i.e. the spherical shell) can be found from

$$\overline{\chi_s \tilde{\mathbf{T}}_s} = \frac{1}{\frac{4}{3}\pi R^3} \int\int\int_{V_s} \tilde{\mathbf{T}}_s \mathrm{d}V, \tag{63}$$

where V_s is the volume of the shell.

Using Eq. (8), we may rewrite the stress tensor of the shell as

$$\tilde{\mathbf{T}}_s = \nabla \cdot (\tilde{\mathbf{T}}_s \mathbf{x}'). \tag{64}$$

If we insert Eq. (64) into (63), we obtain

$$\overline{\chi_s \tilde{\mathbf{T}}_s} = \frac{1}{\frac{4}{3}\pi R^3} \int\int\int_{V_s} \nabla \cdot (\tilde{\mathbf{T}}_s \mathbf{x}')\, \mathrm{d}V. \tag{65}$$

Applying the Gauss theorem

$$\overline{\chi_s \tilde{\mathbf{T}}_s} = \frac{1}{\frac{4}{3}\pi R^3} \int\int_{\Omega(a)} \mathbf{n}_L \cdot (\tilde{\mathbf{T}}_s \mathbf{x}')_{ex} \mathrm{d}\Omega(a) + \frac{1}{\frac{4}{3}\pi R^3} \int\int_{\Omega(a)} \mathbf{n}_G \cdot (\tilde{\mathbf{T}}_s \mathbf{x}')_{in} \mathrm{d}\Omega(a). \tag{66}$$

The normal components of the stress at the surfaces of the shell can be taken to be:

$$\mathbf{n}_L \cdot (\tilde{\mathbf{T}}_s)_{ex} = \mathbf{n}_L \cdot \tilde{\mathbf{T}}_{Li} = -\tilde{p}_{Li} \mathbf{n}_L \tag{67}$$

$$\mathbf{n}_G \cdot (\tilde{\mathbf{T}}_s)_{in} = \mathbf{n}_G \cdot \tilde{\mathbf{T}}_{Gi} = -p_{Gi} \mathbf{n}_G, \tag{68}$$

where the normal component of the phasic viscous shears are assumed small compared to the phasic pressure. Thus, Eq. (66) becomes:

$$\overline{\chi_s \tilde{\mathbf{T}}_s} \cong \frac{1}{\frac{4}{3}\pi R^3} \int\int_{\Omega(a)} \tilde{p}_{Li} \mathbf{n}_L \mathbf{x}' \mathrm{d}\Omega(a) - \frac{1}{\frac{4}{3}\pi R^3} \int\int_{\Omega(a)} \tilde{p}_{Gi} \mathbf{n}_G \mathbf{x}' \mathrm{d}\Omega(a). \tag{69}$$

Evaluating the integral in Eq. (69) using the interfacial pressure of the continuous phase given by Eq. (39), we obtain,

$$\overline{\chi_s \tilde{\mathbf{T}}_s} = \epsilon_G \left[-\frac{9}{20} \rho_L \mathbf{v}_r \mathbf{v}_r + \left(\frac{8}{20} \rho_L \mathbf{v}_r \cdot \mathbf{v}_r - (\rho_{Gi} - \rho_{Li}) \right) \mathbf{I} \right], \tag{70}$$

where it has been assumed that,

$$p_G = p_{Gi} = \text{constant}.$$

Alternatively, we can rewrite Eq. (70) using Eq. (41) as:

$$\overline{\nabla \cdot (\chi_s \mathbf{T}_s)} = \nabla \cdot [\epsilon_G (\sigma_s + (p_{Gi} - p_{Li}) \mathbf{I})], \tag{71}$$

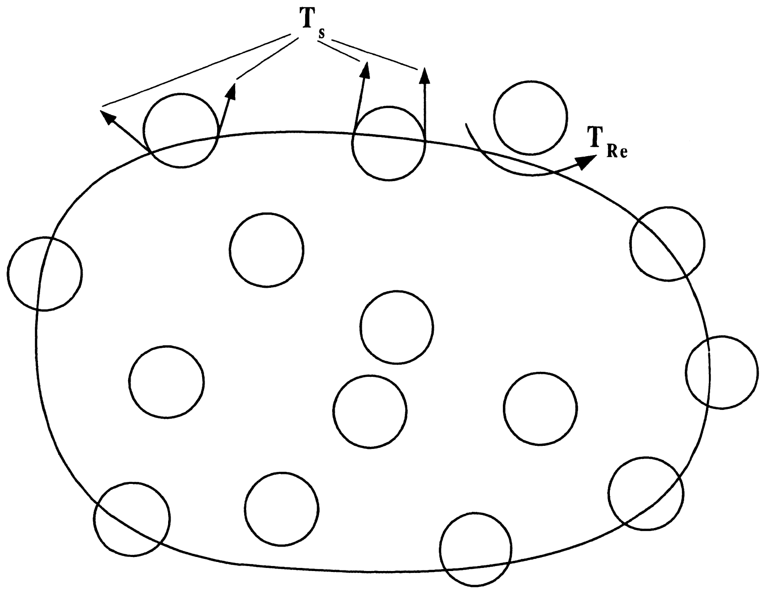

Fig. 2. The source of interfacial stresses.

where

$$\sigma_s = \rho_L\left[-\frac{9}{20}\mathbf{v}_r\mathbf{v}_r + \frac{3}{20}(\mathbf{v}_r\cdot\mathbf{v}_r)\mathbf{I}\right]. \tag{72}$$

The source of the average interfacial stress, σ_s, is illustrated schematically in Fig. 2. We see that an average interfacial stress may arise due to a net unbalanced interfacial force when the interface is cut by the control volume.

Thus, we obtain the interfacial momentum source for the dispersed phase, using the interfacial momentum jump condition, Eq. (18) and Eqs. (51) and (71), as,

$$\mathbf{M}_{Gi} = -\mathbf{M}_{Li} + [\epsilon_G(\sigma_s + (p_{Gi} - p_{Li})\mathbf{I})]$$

or, partitioning into drag and non-drag components,

$$\mathbf{M}_{Gi} = -\mathbf{M}_{Li}^{(nd)} - \mathbf{M}_{Li}^{(d)} + (\tau_{Li} + \tau_{Li}^{Re})\cdot\nabla\epsilon_L + \nabla\cdot(\epsilon_G\sigma_s) + \nabla(\epsilon_G p_{Gi}) - \epsilon_G\nabla p_{Li}. \qquad (73]$$

4. Averaged conservation equations

Let us now summarize the averaged conservation equations derived in the previous section, for adiabatic bubbly flows,

Mass balance:

$$\frac{\partial}{\partial t}(\epsilon_L \rho_L) + \nabla \cdot (\epsilon_L \rho_L \mathbf{v}_L) = 0 \tag{74}$$

$$\frac{\partial}{\partial t}(\epsilon_G \rho_G) + \nabla \cdot (\epsilon_G \rho_G \mathbf{v}_G) = 0. \tag{75}$$

Momentum balance:

$$\frac{\partial}{\partial t}(\epsilon_L \rho_L \mathbf{v}_L) + \nabla \cdot (\epsilon_L \rho_L \mathbf{v}_L \mathbf{v}_L) = -\epsilon_L \nabla p_L + \Delta p_{Li} \nabla \epsilon_L + \nabla \cdot [\epsilon_L(\tau_L + \tau_L^{Re})] + \epsilon_L \rho_L \mathbf{g} - (\tau_{Li} + \tau_{Li}^{Re}) \cdot \nabla \epsilon_L + \mathbf{M}_{Li}^{(nd)} + \mathbf{M}_{Li}^{(d)} \tag{76}$$

$$\frac{\partial}{\partial t}(\epsilon_G \rho_G \mathbf{v}_G) + \nabla \cdot (\epsilon_G \rho_G \mathbf{v}_G \mathbf{v}_G) = -\epsilon_G \nabla p_G - (\tau_{Li} + \tau_{Li}^{Re}) \cdot \nabla \epsilon_G + \nabla \cdot [\epsilon_G(\tau_G + \tau_G^{Re} + \sigma_s)] + \epsilon_G \rho_G \mathbf{g} - \mathbf{M}_{Li}^{(nd)} - \mathbf{M}_{Li}^{(d)}, \tag{77}$$

where

$$p_L = p_o - \frac{1}{4} \epsilon_G \rho_L \mathbf{v}_r \cdot \mathbf{v}_r \tag{78}$$

$$p_{Li} = p_o - \frac{1}{4} \rho_L \mathbf{v}_r \cdot \mathbf{v}_r = p_G \tag{79}$$

$$\Delta p_{Li} = -\frac{1}{4} \rho_L \epsilon_L \mathbf{v}_r \cdot \mathbf{v}_r \tag{80}$$

$$\tau_L^{Re} = -\frac{1}{20} \rho_L \frac{\epsilon_G}{\epsilon_L} [\mathbf{v}_r \mathbf{v}_r + 3(\mathbf{v}_r \cdot \mathbf{v}_r)\mathbf{I}] \tag{81}$$

$$\tau_{Li}^{Re} = -\frac{1}{20} \rho_L [\mathbf{v}_r \mathbf{v}_r + 3(\mathbf{v}_r \cdot \mathbf{v}_r)\mathbf{I}] \tag{82}$$

$$\mathbf{M}_{\mathrm{Li}}^{(\mathrm{nd})} = \frac{1}{2}\epsilon_G\rho_L\mathbf{a}_{\mathrm{vm}} + \frac{1}{4}\epsilon_G\rho_L\mathbf{v}_r\cdot(\nabla\mathbf{v}_L^T - \nabla\mathbf{v}_L) + \frac{1}{2}\rho_L\mathbf{v}_r\left(\frac{D_G\epsilon_G}{Dt} + \epsilon_G\nabla\cdot\mathbf{v}_G\right) + \frac{5}{4}\epsilon_G\rho_L\mathbf{v}_r\cdot\nabla\mathbf{v}_r^T - \frac{9}{20}\epsilon_G\rho_L[\mathbf{v}_r\cdot(\nabla\mathbf{v}_r + \nabla\mathbf{v}_r^T) + (\nabla\cdot\mathbf{v}_r)\mathbf{v}_r] + \frac{3}{10}\rho_L(\mathbf{v}_r\cdot\mathbf{v}_r\nabla\epsilon_G) - \frac{2}{5}\rho_L(\mathbf{v}_r\cdot\nabla\epsilon_G)\mathbf{v}_r + C_L\rho_L\epsilon_G\mathbf{v}_r\times\nabla\times\mathbf{v}_L \tag{83}$$

$$\mathbf{M}_{\mathrm{Li}}^{(\mathrm{d})} = \frac{3}{8}\frac{C_D}{R_b}\rho_L\epsilon_G\mid\mathbf{v}_r\mid\mathbf{v}_r \tag{84}$$

$$\mathbf{a}_{\mathrm{vm}} = \frac{D_G\mathbf{v}_G}{Dt} - \frac{D_L\mathbf{v}_L}{Dt} \tag{85}$$

$$\sigma_s = \rho_L\left[-\frac{9}{20}\mathbf{v}_r\mathbf{v}_r + \frac{3}{20}(\mathbf{v}_r\cdot\mathbf{v}_r)\mathbf{I}\right] \tag{86}$$

$$\mathbf{v}_r = \mathbf{v}_G - \mathbf{v}_L \tag{87}$$

and $\tau_{\mathrm{Li}} = \tau_G^{Re} = \tau_G = 0$.

Combining Eqs. (78)–(87) with Eqs. (76) and (77) and neglecting the viscous stresses (i.e. τ_L and τ_G), we obtain the phasic momentum equations as,

$$\frac{\partial}{\partial t}(\epsilon_L\rho_L\mathbf{v}_L) + \nabla\cdot(\epsilon_L\rho_L\mathbf{v}_L\mathbf{v}_L) = -\epsilon_L\nabla p_L + \Delta p_{\mathrm{Li}}\nabla\epsilon_L + \frac{1}{2}\epsilon_G\rho_L\mathbf{a}_{\mathrm{vm}} + \frac{1}{4}\epsilon_G\rho_L\mathbf{v}_r\cdot(\nabla\mathbf{v}_G^T - \nabla\mathbf{v}_G) + \frac{1}{2}\rho_L\mathbf{v}_r\left(\frac{D_G\epsilon_G}{Dt} + \epsilon_G\nabla\cdot\mathbf{v}_G\right) + \frac{1}{2}\rho_L\epsilon_G\mathbf{v}_r\cdot\nabla\mathbf{v}_r^T - \frac{1}{2}\rho_L\nabla\cdot(\epsilon_G\mathbf{v}_r\mathbf{v}_r) + C_L\rho_L\epsilon_G\mathbf{v}_r\times\nabla\times\mathbf{v}_L + \epsilon_L\rho_L\mathbf{G} + \frac{3}{8}\frac{C_D}{R_b}\rho_L\epsilon_G\mathbf{v}_r\mid\mathbf{v}_r\mid \tag{88}$$

$$\frac{\partial}{\partial t}(\epsilon_G\rho_G\mathbf{v}_G) + \nabla\cdot(\epsilon_G\rho_G\mathbf{v}_G\mathbf{v}_G) = -\epsilon_G\nabla p_{\mathrm{Li}} - \frac{1}{2}\epsilon_G\rho_L\mathbf{a}_{\mathrm{vm}} - \frac{1}{4}\epsilon_G\rho_L\mathbf{v}_r\cdot(\nabla\mathbf{v}_G^T - \nabla\mathbf{v}_G) - \frac{1}{2}\rho_L\mathbf{v}_r\left(\frac{D_G\epsilon_G}{Dt} + \epsilon_G\nabla\cdot\mathbf{v}_G\right) - \frac{1}{2}\epsilon_G\rho_L\mathbf{v}_r\cdot\nabla\mathbf{v}_r^T - C_L\rho_L\epsilon_G\mathbf{v}_r\times\nabla\times\mathbf{v}_L + \epsilon_G\rho_G\mathbf{g} - \frac{3}{8}\frac{C_D}{R_b}\rho_L\epsilon_G\mathbf{v}_r\mid\mathbf{v}_r\mid \tag{89}$$

where terms of higher-order in ϵ_G have been neglected.

The mixture momentum equation can be found by adding Eqs. (76) and (77) as,

$$\frac{\partial}{\partial t}(\epsilon_L\rho_L\mathbf{v}_L+\epsilon_G\rho_G\mathbf{v}_G)+\nabla\cdot(\epsilon_G\rho_G\mathbf{v}_G\mathbf{v}_G+\epsilon_G\rho_G\mathbf{v}_G\mathbf{v}_G)$$
$$=-\epsilon_L\nabla p_L+p_{Li}\nabla\epsilon_L-\epsilon_G\nabla p_{Li}+\nabla\cdot(\epsilon_L\tau_G^{Re}+\epsilon_G\sigma_s)$$
$$+(\epsilon_G\rho_G+\epsilon_G\rho_G)\mathbf{g}, \tag{90}$$

where τ_L, τ_G and τ_G^{Re} are neglected.

Noting that from Eqs. (81) and (86),

$$\epsilon_L\tau_L^{Re}+\epsilon_G\sigma_s=-\frac{1}{2}\epsilon_G\rho_L\mathbf{v}_r\mathbf{v}_r \tag{91}$$

we may rearrange Eq. (90) as:

$$\frac{\partial}{\partial t}(\epsilon_L\rho_L\mathbf{v}_L+\epsilon_G\rho_G\mathbf{v}_G)+\nabla\cdot(\epsilon_L\rho_L\mathbf{v}_L\mathbf{v}_L+\epsilon_G\rho_G\mathbf{v}_G\mathbf{v}_G+\frac{1}{2}\epsilon_G\rho_G\mathbf{v}_r\mathbf{v}_r)$$
$$=-\nabla(\epsilon_Lp_L+\epsilon_Gp_G)+(\epsilon_G\rho_G+\epsilon_G\rho_L)\mathbf{g}. \tag{92}$$

It should be noted that to the first order in ϵ_G, which is in order of accuracy of cell averaging, Eq. (92) is exactly the same as the mixture momentum equation derived by Wallis (1991). Moreover, the $1/2\nabla\cdot(\epsilon_G\rho_L\mathbf{v}_r\mathbf{v}_r)$ term on the left hand side of Eq. (92) arises due to the interfacial stress and the continuous phase Reynolds stress, as noted in Eq. (91).

It is also interesting to compare the two-fluid model just derived to one obtained from a variational approach. Guerst (1986) derived a set of two-fluid equations using a variational principle. His phasic momentum equations were:

$$\frac{\partial}{\partial t}(\rho_L\epsilon_L\mathbf{v}_L)+\nabla\cdot[\rho_L\epsilon_L\mathbf{v}_L\mathbf{v}_L+\rho_Lm(\epsilon_G)\mathbf{v}_r\mathbf{v}_r]$$
$$+\epsilon_L\nabla p_G+\nabla\left[\frac{1}{2}\rho_L(m(\epsilon_G)+\epsilon_Lm'(\epsilon_G))\mathbf{v}_r\cdot\mathbf{v}_r\right]=\mathbf{M}^G \tag{93}$$

$$\frac{\partial}{\partial t}(\rho_G\epsilon_G\mathbf{v}_G)+\nabla\cdot(\rho_G\epsilon_G\mathbf{v}_G\mathbf{v}_G)+\epsilon_G\nabla p_G=-\mathbf{M}^G, \tag{94}$$

where

$$\mathbf{M}^G=\frac{\partial}{\partial t}[\rho_Lm(\epsilon_G)\mathbf{v}_r]+\nabla\cdot[\rho_Lm(\epsilon_G)\mathbf{v}_G\mathbf{v}_r]+\rho_Lm(\epsilon_G)\mathbf{v}_r\cdot\nabla\mathbf{v}_g^T, \tag{95}$$

If we rearrange Eqs. (88) and (89), we obtain:

$$\frac{\partial}{\partial t}(\epsilon_L\rho_L\mathbf{v}_L)+\nabla\cdot(\epsilon_L\rho_L\mathbf{v}_L\mathbf{v}_L+\frac{1}{2}\rho_L\epsilon_G\mathbf{v}_r\mathbf{v}_r)+\epsilon_L\nabla p_L-\Delta p_{Li}\nabla\epsilon_L=\mathbf{M} \tag{96}$$

$$\frac{\partial}{\partial t}(\epsilon_G\rho_G\mathbf{v}_G)+\nabla\cdot(\epsilon_G\rho_G\mathbf{v}_G\mathbf{v}_G)+\epsilon_G\nabla p_{Li}=-\mathbf{M} \tag{97}$$

where

$$\mathbf{M} \triangleq \frac{1}{2}\epsilon_G\rho_L\mathbf{a}_{vm} + \frac{1}{4}\epsilon_G\rho_L\mathbf{v}_r\cdot(\nabla\mathbf{v}_G^T - \nabla\mathbf{v}_G)$$

$$+\frac{1}{2}\rho_L\mathbf{v}_r\left(\frac{D_g\epsilon_G}{Dt} + \epsilon_G\nabla\cdot\mathbf{v}_G\right) + \frac{1}{2}\rho_L\epsilon_G\mathbf{v}_r\cdot\nabla\mathbf{v}_r^T$$

$$+C_L\rho_L\epsilon_G\mathbf{v}_r\times\nabla\times\mathbf{v}_L. \tag{98}$$

To facilitate comparison with Guerst's result, we may rewrite Eq. (98) as:

$$\mathbf{M} = \frac{1}{2}\frac{\partial}{\partial t}(\rho_L\epsilon_G\mathbf{v}_r) + \frac{1}{2}\nabla\cdot(\rho_L\epsilon_G\mathbf{v}_G\mathbf{v}_r) + \frac{1}{2}\rho_L\epsilon_G\mathbf{v}_r\cdot\nabla\mathbf{v}_G^T$$

$$+\frac{1}{4}\epsilon_G\rho_L\mathbf{v}_r\cdot(\nabla\mathbf{v}_r^T - \nabla\mathbf{v}_r) + \frac{1}{2}\rho_L\mathbf{v}_r\left(\frac{D_G\epsilon_G}{Dt} + \epsilon_G\nabla\cdot\mathbf{v}_r\right) \tag{99}$$

where $C_L = 1/4$ has been used.

If we note that $m(\epsilon_G) = 1/2\epsilon_G$ for the virtual volume in Eq. (95), we find that the discrepancy between the two dispersed phase momentum equations consists of the last two terms in Eq. (99).

We note that for incompressible bubbles, we have

$$\frac{D_G\epsilon_G}{Dt} + \epsilon_G\nabla\cdot\mathbf{v}_G = 0. \tag{100}$$

Thus, the discrepancy is only in the term

$$\mathbf{M}_{rot} = \frac{1}{4}\epsilon_G\rho_L\mathbf{v}_r\cdot(\nabla\mathbf{v}_r^T - \nabla\mathbf{v}_r).$$

We note that the rate at which $\mathbf{M}_{rot}$ does work is

$$\mathbf{v}_r\cdot\mathbf{M}_{rot} = \frac{1}{4}\epsilon_G\rho_L(\mathbf{v}_r\cdot(\nabla\mathbf{v}_r^T - \nabla\mathbf{v}_r))\cdot\mathbf{v}_r = 0. \tag{101}$$

Therefore, work done by a term such as $\mathbf{M}_{rot}$ cannot appear in the kinetic energy equation, and hence cannot appear in the variational equations derived from it. The equations derived from it are not unique, module terms of the form $\mathbf{M}_{rot}$.

If we assume that the phasic densities are constant, we obtain, by adding Eqs. (74) and (75),

$$\nabla\cdot\mathbf{j} = 0, \tag{102}$$

where the volumetric flux is defined as,

$$\mathbf{j} \triangleq \epsilon_L\mathbf{v}_L + \epsilon_G\mathbf{v}_G. \tag{103}$$

One way to assess the validity of the two-fluid model derived herein is to examine the behavior of the governing equations in limiting cases. To this end, let us consider the situation in the limit where the average volume fraction of the dispersed phase, ϵ_G, vanishes.

Thus, if we neglect ϵ_G and $\nabla\epsilon_G$ in Eqs. (102) and (103), we obtain

$$\nabla \cdot \mathbf{v}_L = 0, \tag{104}$$

which agrees with the continuity equation of single-phase incompressible flows. For this limiting case, the continuous phase's momentum equation, Eq. (88), yields:

$$\rho_L \frac{D_L \mathbf{v}_L}{Dt} = -\nabla p_L + \rho_L \mathbf{g}. \tag{105}$$

It should be noted that Eq. (105) correctly yields the momentum equation of single-phase inviscid flow.

We can simplify Eq. (89) by expanding the material derivative of the discontinuous volume fraction as

$$\frac{D_G \epsilon_G}{Dt} = \frac{\partial \epsilon_G}{\partial t} + \mathbf{v}_G \cdot \nabla \epsilon_G = -\epsilon_G \nabla \cdot \mathbf{v}_G, \tag{106}$$

where Eq. (75) has been used.

Using Eqs. (78)–(80) and (106), we obtain a simplified form of Eq. (89) as:

$$\rho_G \frac{D_G \mathbf{v}_G}{Dt} = -\nabla p_L - \frac{1}{2}\rho_L\left(\frac{D_G \mathbf{v}_G}{Dt} - \frac{D_L \mathbf{v}_L}{Dt}\right) - \frac{1}{4}\rho_L \mathbf{v}_r \times \nabla \times \mathbf{v}_r$$

$$-\frac{1}{2}\rho_L \mathbf{v}_r \times \nabla \times \mathbf{v}_L + \rho_G \mathbf{g}. \tag{107}$$

Let us now consider the case of a single bubble rising in a stagnant pool of liquid. If the radius of the pool is very large, we may neglect the average volume fraction of the discontinuous phase and its derivative. Thus, by combining Eqs. (105), (107) and (104) with $\mathbf{v}_L = 0$, we obtain,

$$\left(\rho_G + \frac{1}{2}\rho_L\right)\frac{D_G \mathbf{v}_G}{Dt} = -(\rho_L - \rho_G)\mathbf{g}, \tag{108}$$

where the lateral lift forces are zero in this case since $\nabla\times\ \mathbf{v}_G = \nabla\times\ \mathbf{v}_L = \mathbf{0}$. Eq. (108) is a well known result for inviscid flows.

Another interesting case is that of a single bubble placed in a large horizontal converging stream of liquid. If we somehow apply an external force, $\mathbf{F}_{ex}$, to hold the bubble fixed, we obtain from Eqs. (105) and (107),

$$\rho_L \mathbf{v}_L \cdot \nabla \mathbf{v}_L = -\nabla p_L \tag{109}$$

$$0 = -\nabla p_L + \frac{1}{2}\rho_L \mathbf{v}_L \cdot \nabla \mathbf{v}_L - \mathbf{F}_{ex}. \tag{110}$$

Combining Eqs. (109) and (110), we obtain the force necessary to fix the bubble as:

$$\mathbf{F}_{\mathrm{ex}} = \left(1 + \frac{1}{2}\right)\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{L}} \cdot \nabla\mathbf{v}_{\mathrm{L}} = -\left(1 + \frac{1}{2}\right)p_{\mathrm{L}} = -\frac{3}{2}\nabla p_{\mathrm{L}}. \tag{111}$$

This agrees with the result previously obtained by Taylor (1924) for a spherical particle.

It was pointed out by Wallis (1991) that some averaged two-fluid models have an inconsistency in predicting $\mathbf{F}_{\mathrm{ex}}$. For example, models used by Pauchon and Banerjee (1986), Arnold (1988) and Park et al. (1994) resulted in:

$$\mathbf{F}_{\mathrm{ex}} = \left(1 + \frac{1}{4} + \frac{1}{4}\right)\rho_{\mathrm{L}}\mathbf{v}_{\mathrm{L}} \cdot \nabla\mathbf{v}_{\mathrm{L}} = -2\nabla p_{\mathrm{L}}. \tag{112}$$

The reason for this discrepancy is that these two-fluid models did not include the surface stress developed on the dispersed particles to maintain the spherical shape of the particle. In this study, this inconsistency has been removed by properly including the average interfacial stress (σ_{s}) in the two-fluid model.

5. One-dimensional conservation equations and their characteristics

The one-dimensional form of the mass and the momentum conservation equations can be obtained by considering the z-directional component of the phasic velocities and forces in Eqs. (74)–(87) as:

$$\frac{\partial}{\partial t}(\epsilon_{\mathrm{L}}\rho_{\mathrm{L}}) + \frac{\partial}{\partial z}(\epsilon_{\mathrm{L}}\rho_{\mathrm{L}}u_{\mathrm{L}}) = 0 \tag{113}$$

$$\frac{\partial}{\partial t}(\epsilon_{\mathrm{G}}\rho_{\mathrm{G}}) + \frac{\partial}{\partial z}(\epsilon_{\mathrm{G}}\rho_{\mathrm{G}}u_{\mathrm{G}}) = 0 \tag{114}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\epsilon_{\mathrm{L}}\rho_{\mathrm{L}}u_{\mathrm{L}}) + \frac{\partial}{\partial z}(\epsilon_{\mathrm{L}}\rho_{\mathrm{L}}u_{\mathrm{L}}^2) &= -\epsilon_{\mathrm{L}}\frac{\partial p_{\mathrm{L}}}{\partial z} + \Delta p_{\mathrm{Li}}\frac{\partial \epsilon_{\mathrm{L}}}{\partial z} - \tau_{\mathrm{Li}}^{Re}\frac{\partial \epsilon_{\mathrm{L}}}{\partial z} \\ &\quad + \frac{\partial}{\partial z}[\epsilon_{\mathrm{L}}(\tau_{\mathrm{L}} + \tau_{\mathrm{L}}^{Re})] + \epsilon_{\mathrm{L}}\rho_{\mathrm{L}}g\cos\theta + M_{\mathrm{Li}}^{(\mathrm{nd})} + M_{\mathrm{Li}}^{(\mathrm{d})} - 4\frac{\tau_{\mathrm{Lw}}}{D_{\mathrm{H}}}\end{aligned} \tag{115}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\epsilon_{\mathrm{G}}\rho_{\mathrm{G}}u_{\mathrm{G}}) + \frac{\partial}{\partial z}(\epsilon_{\mathrm{G}}\rho_{\mathrm{G}}u_{\mathrm{G}}^2) &= -\epsilon_{\mathrm{G}}\frac{\partial p_{\mathrm{Li}}}{\partial z} - \tau_{\mathrm{Li}}^{Re}\frac{\partial \epsilon_{\mathrm{G}}}{\partial z} \\ &\quad + \frac{\partial}{\partial z}[\epsilon_{\mathrm{G}}(\tau_{\mathrm{G}} + \sigma_{\mathrm{s}})] + \epsilon_{\mathrm{G}}\rho_{\mathrm{G}}g\cos\theta - M_{\mathrm{Li}}^{(\mathrm{nd})} - M_{\mathrm{Li}}^{(\mathrm{d})} - 4\frac{\tau_{\mathrm{Gw}}}{D_{\mathrm{H}}},\end{aligned} \tag{116}$$

where θ denotes the angle between the axial direction, z, and the gravity vector.

The closure relations for this one-dimensional system of equations can be summarized as follows:

$$M_{\mathrm{Li}}^{(\mathrm{nd})} = C_{\mathrm{vm}}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}a_{\mathrm{vm}} - C_{\mathrm{m1}}\epsilon_{\mathrm{G}}\rho_{\mathrm{L}}u_{\mathrm{r}}\frac{\partial u_{\mathrm{r}}}{\partial z} - C_{\mathrm{m2}}\rho_{\mathrm{L}}u_{\mathrm{r}}^2\frac{\partial \epsilon_{\mathrm{G}}}{\partial z} \tag{117}$$

$$M_{Li}^{(d)} = \frac{3}{8}\frac{C_D}{R_b}\rho_L\epsilon_G \mid u_r \mid u_r \tag{118}$$

$$p_L = p_o - C_p\rho_L\epsilon_G u_r^2 \tag{119}$$

$$\Delta p_{Li} = p_{Li} - p_L = -C_p\rho_L\epsilon_L u_r^2 \tag{120}$$

$$\sigma_s = -C_i\rho_L u_r^2 \tag{121}$$

$$\tau_L^{Re} = -C_r\rho_L\frac{\epsilon_G}{\epsilon_L} \mid u_r \mid u_r \tag{122}$$

$$\tau_{Li}^{Re} = -C_r\rho_L u_r^2 \tag{123}$$

$$\tau_{Lw} = \frac{1}{2}f_{Lw}\rho_L u_L \mid u_L \mid \tag{124}$$

$$\tau_{Gw} = \frac{1}{2}f_{Gw}\rho_G u_G \mid u_G \mid, \tag{125}$$

where, p_o would be the local liquid phase pressure if the sphere were not present, and:

$$a_{vm} = \frac{\partial u_G}{\partial t} + u_G\frac{\partial u_G}{\partial z} - \left(\frac{\partial u_L}{\partial t} + u_L\frac{\partial u_L}{\partial z}\right) \tag{126}$$

$$u_r = u_G - u_L \tag{127}$$

$$C_{vm} = \frac{1}{2}, \quad C_p = \frac{1}{4}, \quad C_r = \frac{1}{5}, \quad C_i = \frac{3}{10} \quad C_{m1} = C_{m2} = \frac{1}{10}. \tag{128}$$

Note that the viscous stresses, τ_L and τ_G, have been neglected.

If we insert the constitutive relations given by Eqs. (117)–(125) into Eqs. (115) and (116), we obtain:

$$\rho_L\epsilon_L\frac{D_L u_L}{Dt} = -\epsilon_L\frac{\partial p_L}{\partial z} - C_p\rho_L\epsilon_L u_r^2\frac{\partial\epsilon_L}{\partial z} + C_r\rho_L u_r^2\frac{\partial\epsilon_L}{\partial z} - \frac{\partial}{\partial z}(C_r\rho_L\epsilon_G u_r^2) + \epsilon_L\rho_L g\cos\theta + C_{vm}\epsilon_G\rho_L a_{vm} - C_{m1}\epsilon_G\rho_L u_r\frac{\partial u_r}{\partial z} - C_{m2}\rho_L u_r^2\frac{\partial\epsilon_G}{\partial z} + \frac{3}{8}\epsilon_G\frac{C_D}{R_b}\rho_L u_r^2 - 2\epsilon_L\rho_L\frac{f_{Lw}}{D_H}u_L \mid u_L \mid \tag{129}$$

$$\rho_G\epsilon_G\frac{D_G u_G}{Dt} = -\epsilon_G\frac{\partial p_L}{\partial z} + \epsilon_G\frac{\partial}{\partial z}(C_p\rho_L\epsilon_L u_r^2) + C_r\rho_L u_r^2\frac{\partial \epsilon_G}{\partial z}$$
$$-\frac{\partial}{\partial z}(C_i\rho_L\epsilon_G u_r^2) + \epsilon_G\rho_G g\cos\theta - C_{vm}\epsilon_G\rho_L a_{vm} + C_{m1}\epsilon_G\rho_L u_r\frac{\partial u_r}{\partial z}$$
$$+C_{m2}\rho_L u_r^2\frac{\partial\epsilon_G}{\partial z} - \frac{3}{8}\epsilon_G\frac{C_D}{R_b}\rho_L u_r^2 - 2\epsilon_G\rho_G\frac{f_{Gw}}{D_H}u_G \mid u_G \mid . \tag{130}$$

To eliminate the pressure gradients, we divide Eqs. (129) and (130) by $\rho_L\epsilon_L$ and $\rho_L\epsilon_G$ respectively, and subtract one from the other:

$$\left(\rho_G^* + \frac{C_{vm}}{\epsilon_L}\right)\frac{D_G u_G}{Dt} - \left(1 + \frac{C_{vm}}{\epsilon_L}\right)\frac{D_L u_L}{Dt} = 2\left(C_p\epsilon_L + C_r\frac{\epsilon_G}{\epsilon_L} - C_i + \frac{C_{m1}}{2\epsilon_L}\right)u_r\frac{\partial u_r}{\partial z}$$
$$-\left(2C_p - \frac{2\epsilon_G+\epsilon_L}{\epsilon_G\epsilon_L}C_r + \frac{C_i}{\epsilon_G} - \frac{C_{m2}}{\epsilon_L\epsilon_G}\right)u_r^2\frac{\partial\epsilon_G}{\partial z}$$
$$-\frac{3}{8}\frac{C_D}{\epsilon_L R_b}u_r^2 + (f_{Lw} \mid u_L \mid u_L - f_{Gw} \mid u_G \mid u_G) \tag{131}$$

where $\rho_G^* = \rho_G/\rho_L$.

If we rewrite Eqs. (113), (114) and (131) in matrix form, we obtain the one-dimensional system of equation as,

$$\mathbf{A}\frac{\partial\Phi}{\partial t} + \mathbf{B}\frac{\partial\Phi}{\partial z} = \mathbf{c} \tag{132}$$

where

$$\Phi \triangleq \begin{pmatrix} \epsilon_G \\ u_G \\ u_L \end{pmatrix}$$

$$\mathbf{c} \triangleq \begin{pmatrix} 0 \\ 0 \\ c_o \end{pmatrix}$$

$$\mathbf{A} \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \rho_G^* + \frac{C_{vm}}{\epsilon_L} & -\left(1+\frac{C_{vm}}{\epsilon_L}\right) \end{pmatrix}$$

$$\mathbf{B} \triangleq \begin{pmatrix} u_G & \epsilon_G & 0 \\ u_L & 0 & -\epsilon_L \\ B_1 u_r^2 & \left(\rho_G^* + \frac{C_{vm}}{\epsilon_L}\right)u_G - B_2 u_r & -\left(1+\frac{C_{vm}}{\epsilon_L}\right)u_L + B_2 u_r \end{pmatrix}$$

$$B_1 \triangleq 2C_p - \frac{2\epsilon_G + \epsilon_L}{\epsilon_G\epsilon_L}C_r + \frac{C_i}{\epsilon_G} - \frac{C_{m2}}{\epsilon_L\epsilon_G}$$

$$B_2 \triangleq 2\left(\epsilon_L C_p + \frac{\epsilon_G}{\epsilon_L}C_r - C_i + \frac{C_{m1}}{2\epsilon_L}\right)$$

$$c_o \triangleq -\frac{3}{8}\frac{C_D}{\epsilon_L R_b}u_r \mid u_r \mid + \frac{2}{D_H}(f_{Lw} \mid u_L \mid u_L - \rho_G^* g_{Gw} \mid u_G \mid u_G) + (\rho_G^* - 1)g\cos\theta$$

The system's characteristics (i.e. the eigenvalues, λ) can be found by solving:

$$\det(\mathbf{B} - \lambda\mathbf{A}) = 0. \tag{133}$$

Using the definitions of the system matrices, we obtain the characteristic question as:

$$\epsilon_L(\lambda - u_G)\left[\left(\rho_G^* + \frac{C_{vm}}{\epsilon_L}\right)(\lambda - u_G) + B_2 u_r\right] + \epsilon_G(\lambda - u_L)\left[\left(1 + \frac{C_{vm}}{\epsilon_L}\right)(\lambda - u_L) + B_2 u_r\right] - \epsilon_L\epsilon_G B_1 u_r^2 = 0. \tag{134}$$

If we define,

$$\lambda^* \triangleq \frac{\lambda - u_L}{u_G - u_L}$$

the characteristic equation becomes,

$$a_1\lambda^{*2} + a_2\lambda^* + a_3 = 0, \tag{135}$$

where

$$a_1 = \epsilon_L\left(\rho_G^* + \frac{C_{vm}}{\epsilon_L}\right) + \epsilon_G\left(1 + \frac{C_{vm}}{\epsilon_L}\right)$$

$$a_2 = 2\left[-\epsilon_L\left(\rho_G^* + \frac{C_{vm}}{\epsilon_L}\right) + \frac{B_2}{2}\right]$$

$$a_3 = \epsilon_L\left(\rho_G^* + \frac{C_{vm}}{\epsilon_L}\right) - \epsilon_L\epsilon_G B_1 - \epsilon_L B_2.$$

Solving Eq. (135) for λ^*,

$$\lambda_\pm^* = V^* \pm \sqrt{\upsilon^*/\tau^*}, \tag{136}$$

where

$$V^* = \epsilon_L\frac{[C_{vm} - B_2/2 + \rho_G^*\epsilon_L]}{\epsilon_L\epsilon_G + C_{vm} + \rho_G^*\epsilon_L^2} \tag{137}$$

$$\tau^* = \epsilon_L\epsilon_G + C_{vm} + \rho_G^*\epsilon_L^2 \tag{138}$$

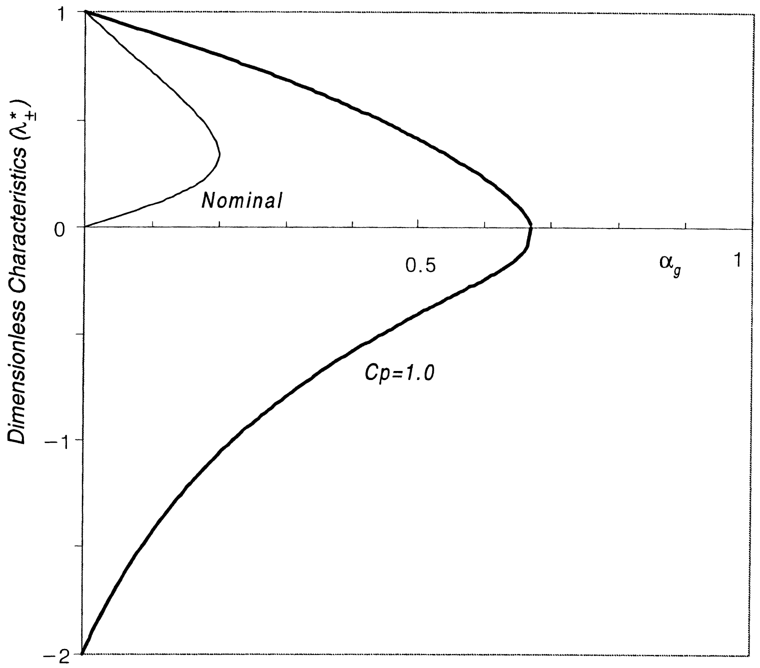

Fig. 3. The effect of C_p on the system's characteristics.

$$\upsilon^* = \epsilon_L^2 \frac{[C_{vm} - B_2/2 + \rho_G^* \epsilon_L]^2}{\epsilon_L \epsilon_G + C_{vm} + \rho_G^* \epsilon_L} - \epsilon_L(\epsilon_L \rho_G^* + C_{vm} - \epsilon_L \epsilon_G B_1 - \epsilon_L B_2). \tag{139}$$

Interestingly, the eigenvalues are always complex for the coefficients given in Eq. (128). However, as can be seen in Fig. 3, the two-fluid model is well-posed in the interval, $0 < \epsilon_G < 0.19$, if we use the following constitutive coefficients: $C_{vm} = 0.5$, $C_p = 0.5$, $C_r = 0.2$, $C_i = 0.3$, $C_{m1} = C_{m2} = 0.1$. It is interesting to note that the well-posed region increases with the increasing values of the interfacial pressure difference associated with helical orbits (e.g. $C_p = 1.0$; Lance and Bataille, 1991). It should be noted that the interfacial pressure difference is a function of two-phase Weber number when the bubble is not spherical (Park and Choi, 1997). Indeed, the characteristic speed with increased values of C_p agrees well with the void wave data as shown in Fig. 6.

As shown in Fig. 4, when fixing $C_p = 0.5$, the system's characteristics also vary with different values of the virtual mass (C_{vm}), the two-phase Reynolds stress (C_r), the interfacial stress (C_i). Interestingly, a reduced value of the virtual volume coefficient (C_{vm}) was found to dramatically increase the well-posed region. The system's characteristics are not very sensitive to the bubble-induced Reynolds stress, which is in contrast to the results of some previous studies (Pauchon and Banerjee, 1988) in which a more simplified two-fluid model was used.

Reduced values of the interfacial stress coefficient (C_i) increases the well-posed region. This implies that interfacial momentum transfer in two-phase flow plays an important role in the transient response of a two-fluid model. It should be noted that the void fraction gradient

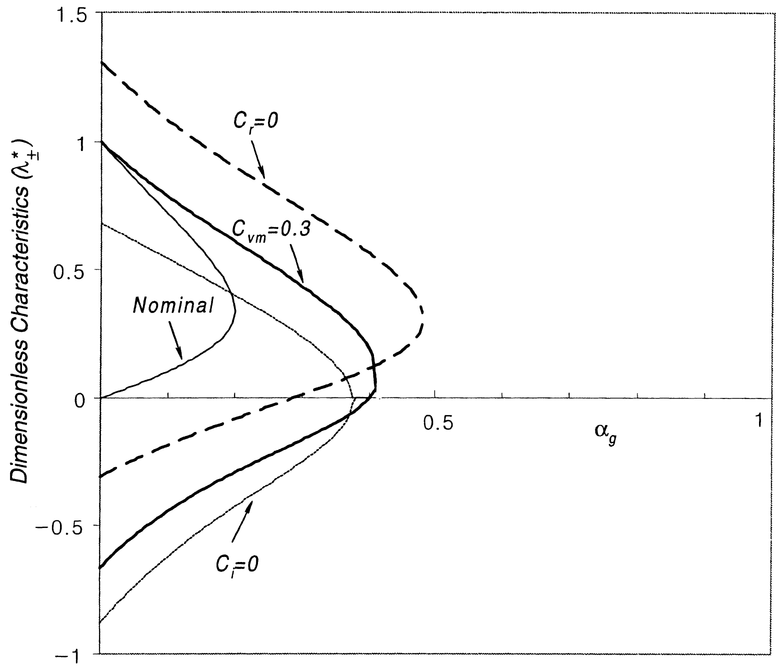

Fig. 4. System's characteristics for different constitutive coefficients.

parameters in the interfacial momentum transfer, C_{m1} and C_{m2} were found to also be important in controlling the system's void wave characteristics.

6. Linear void wave analysis

Let us now investigate the dispersion characteristics of the two-fluid model presented herein. To this end, we introduce a perturbation in two-fluid variables such that,

$$w = w_o + w', \tag{140}$$

where w represents each two-fluid state variable, the subscript o denotes the steady-state and the prime ($'$) a small perturbation.

Inserting Eq. (140) into Eqs. (113)–(116), we obtain the linearized system of conservation equations as:

$$\frac{\partial \epsilon'_L}{\partial t} + u_{Lo}\frac{\partial \epsilon'_L}{\partial z} + \epsilon_{Lo}\frac{\partial u'_L}{\partial z} = 0 \tag{141}$$

$$\frac{\partial \epsilon'_G}{\partial t} + u_{Go}\frac{\partial \epsilon'_G}{\partial z} + \epsilon_{Go}\frac{\partial u'_G}{\partial z} = 0 \tag{142}$$

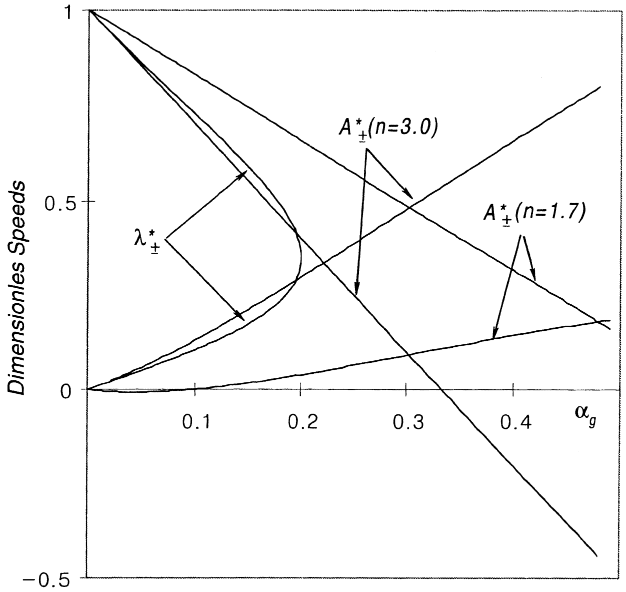

Fig. 5. Kinematic wave speed and characteristics.

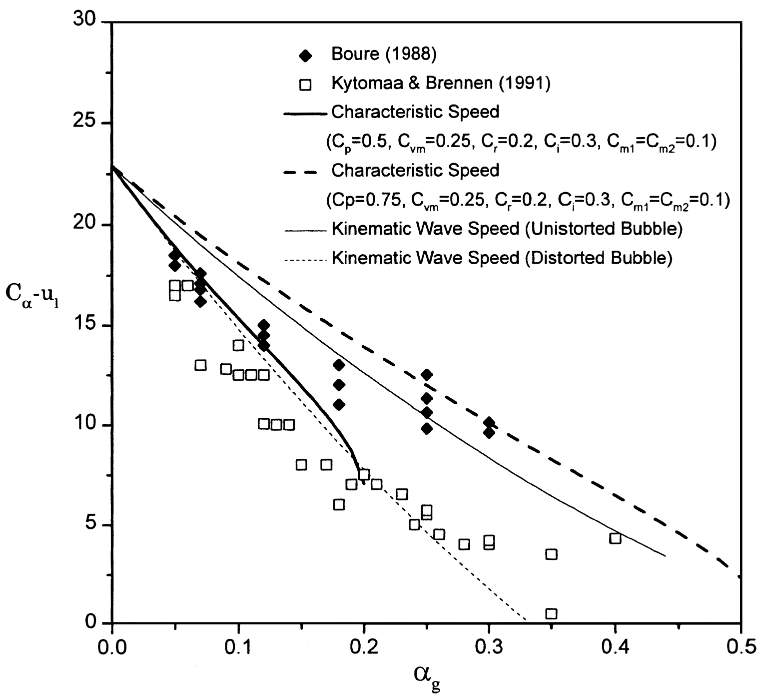

Fig. 6. Characteristics/kinematic void wave speeds and void wave speed data.

$$\rho_{\rm L}\frac{D_{\rm L}u'_{\rm L}}{Dt} = -\frac{\partial p'_{\rm L}}{\partial z} + \frac{\Delta p_{\rm Lio}}{\epsilon_{\rm Lo}}\frac{\partial \epsilon'_{\rm L}}{\partial z} - \frac{\tau^{Re}_{\rm Lio}}{\epsilon_{\rm Lo}}\frac{\partial \epsilon'_{\rm L}}{\partial z} + \frac{\partial}{\partial z}(\tau'_{\rm L} + \tau^{Re}_{\rm L}) + \frac{(\tau_{\rm Lo} + \tau^{Re}_{\rm Lo})}{\epsilon_{\rm Lo}}\frac{\partial \epsilon'_{\rm L}}{\partial z}$$
$$+\frac{(M^{\rm (nd)'}_{\rm Li} + M^{\rm (d)'}_{\rm Li})}{\epsilon_{\rm Go}} + \frac{M^{\rm (d)}_{\rm Lio}}{\epsilon^2_{\rm Lo}}\epsilon'_{\rm L} - \frac{4}{D_{\rm H}}\left(\frac{\tau'_{\rm Lw}}{\epsilon_{\rm Lo}} - \frac{\tau_{\rm Lwo}}{\epsilon^2_{\rm Lo}}\epsilon'_{\rm L}\right) \quad (143)$$

$$\rho_{\rm G}\frac{D_{\rm G}u'_{\rm G}}{Dt} = -\frac{\partial p'_{\rm Li}}{\partial z} - \frac{\tau^{Re}_{\rm Lio}}{\epsilon_{\rm Go}}\frac{\partial \epsilon'_{\rm G}}{\partial z} + \frac{\partial}{\partial z}(\tau'_{\rm G} + \tau^{Re}_{\rm G} + \sigma'_{\rm s}) + \frac{(\tau_{\rm Go} + \tau^{Re}_{\rm Go} + \sigma_{\rm so})}{\epsilon_{\rm Go}}\frac{\partial \epsilon'_{\rm G}}{\partial z}$$
$$-\frac{(M^{\rm (nd)'}_{\rm Li} + M^{\rm (d)'}_{\rm Li})}{\epsilon_{\rm Go}} + \frac{M^{\rm (d)}_{\rm Lio}}{\epsilon^2_{\rm Go}}\epsilon'_{\rm G} - \frac{4}{D_{\rm H}}\left(\frac{\tau'_{\rm Gw}}{\epsilon_{\rm Go}} - \frac{\tau_{\rm Gwo}}{\epsilon^2_{\rm Go}}\epsilon'_{\rm Go}\right). \quad (144)$$

If we divide Eqs. (143) and (144) by $\rho_{\rm L}$ and subtract one from the other to eliminate the pressure gradient, we obtain the combined momentum equation as,

$$\rho^*_{\rm G}\frac{D_{\rm Go}u'_{\rm G}}{Dt} - \frac{D_{\rm Lo}u'_{\rm L}}{Dt} = -\frac{1}{\rho_{\rm L}}\frac{\partial \Delta p'_{\rm Li}}{\partial z} - \frac{\Delta p_{\rm Lio}\partial \epsilon'_{\rm L}}{\rho_{\rm L}\epsilon_{\rm Lo}\partial z} - \frac{\tau^{Re}_{\rm Lio}}{\rho_{\rm L}\epsilon_{\rm Go}\epsilon_{\rm Lo}}\frac{\partial \epsilon'_{\rm G}}{\partial z}$$
$$+\frac{1}{\rho_{\rm L}}\frac{\partial}{\partial z}(\tau'_{\rm G} + \tau^{Re'}_{\rm G} + \sigma'_{\rm s} - \tau'_{\rm L} - \tau^{Re'}_{\rm L}) + \frac{(\tau_{\rm Go} + \tau^{Re}_{\rm G} + \sigma_{\rm so})}{\rho_{\rm L}\epsilon_{\rm Go}}\frac{\partial \epsilon'_{\rm G}}{\partial z}$$
$$-\frac{((\tau_{\rm Lo} + \tau^{Re}_{\rm Lo}))}{\rho_{\rm L}\epsilon_{\rm Lo}}\frac{\partial \epsilon'_{\rm L}}{\partial z} - \frac{(M^{\rm (nd)'}_{\rm Li} + M^{\rm (d)'}_{\rm Li})}{\rho_{\rm L}\epsilon_{\rm Go}\epsilon_{\rm Lo}} + \frac{M_{\rm Lio}}{\epsilon^2_{\rm Go}}\epsilon'_{\rm G} + \frac{M_{\rm Lio}}{\epsilon^2_{\rm Lo}}\epsilon'_{\rm L}$$
$$+\frac{4}{\rho_{\rm L}D_{\rm H}}\left(\frac{\tau'_{\rm Lw}}{\epsilon_{\rm Lo}} - \frac{\tau_{\rm Lwo}}{\epsilon^2_{\rm Lo}}\epsilon'_{\rm L} - \frac{\tau'_{\rm Gw}}{\epsilon_{\rm Go}} + \frac{\tau_{\rm Gwo}}{\epsilon^2_{\rm Go}}\epsilon'_{\rm G}\right). \quad (145)$$

To achieve closure, the constitutive relations must be expressed in terms of the state variables, $u_{\rm L}, u_{\rm G}$ and $(\epsilon \triangleq \epsilon_G = 1 - \epsilon_L$. We note that all perturbations on the right hand side of Eq. (145) except those associated with non-drag effects, are of the form:

$$\mathcal{F} = \mathcal{F}_{(\epsilon)}\epsilon' + \mathcal{F}_{(\rm u_L)}u'_{\rm L} + \mathcal{F}_{(\rm u_G)}u'_{\rm G}, \quad (146)$$

where

$$\mathcal{F}_{(\rm v)} \triangleq \left.\frac{\partial \mathcal{F}}{\partial v}\right|_{\rm o}. \quad (147)$$

The non-algebraic constitutive relations must be treated differently. In particular, the perturbed form of the non-drag interfacial momentum exchange term can be found from Eq. (117) as:

$$M^{\rm (nd)'}_{\rm Li} = C_{\rm vm}\epsilon_{\rm Go}\rho_{\rm L}\left(\frac{D_{\rm Go}u'_{\rm G}}{Dt} - \frac{D_{\rm Lo}u'_{\rm L}}{Dt}\right)$$
$$-C_{\rm m1}\epsilon_{\rm Go}\rho_{\rm L}u_{\rm ro}\frac{\partial(u'_{\rm G} - u'_{\rm L})}{\partial z} - C_{\rm m2}\rho_{\rm L}u^2_{\rm ro}\frac{\partial \epsilon'_{\rm G}}{\partial z}. \quad (148)$$

To eliminate the derivatives of the phasic velocity perturbations, we differentiate Eq. (145) and the perturbed continuity equations (i.e. Eqs. (141) and (142)) with respect to time and space, respectively, and combine them. We thus obtain the combined momentum equation as,

$$K_1\frac{\partial\epsilon'}{\partial t}+K_2\frac{\partial\epsilon'}{\partial z}+K_3\frac{\partial^2\epsilon'}{\partial t^2}+K_4\frac{\partial^2\epsilon'}{\partial t\partial z}+K_5\frac{\partial^2\epsilon'}{\partial z^2}=0, \tag{149}$$

where

$$K_1\triangleq\frac{1}{\rho_L\epsilon_o(1-\epsilon_o)}\left(\frac{M^{(d)}_{Lio(u_L)}}{1-\epsilon_o}-\frac{M^{(d)}_{Lio(u_G)}}{\epsilon_o}\right)+\frac{1}{\rho_L D_H}\left[\frac{1}{(1-\epsilon_o)}\left(\frac{\tau_{Lw(u_L)}}{1-\epsilon_o}-\frac{\tau_{Lw(u_G)}}{\epsilon_o}\right)+\frac{1}{\epsilon_o}\left(\frac{\tau_{Gw(u_L)}}{1-\epsilon_o}-\frac{\tau_{Gw(u_G)}}{\epsilon_o}\right)\right]$$

$$K_2\triangleq-\frac{1}{\rho_L\epsilon_o(1-\epsilon_o)}\left(M^{(d)}_{Lio(\epsilon)}+\frac{M^{(d)}_{Lio(u_L)}}{1-\epsilon_o}u_{L_o}-\frac{M^{(d)}_{Lio(u_G)}}{\epsilon_o}u_{G_o}\right)-\frac{M^{(d)}_{Lio}}{\rho_L(1-\epsilon_o)^2\epsilon_o}+\frac{4}{\rho_L D_H(1-\epsilon_o)}\left(\tau_{Lw(\epsilon)}+\frac{\tau_{Lw(u_L)}}{1-\epsilon_o}u_{Lo}-\frac{\tau_{Lw(u_G)}}{\epsilon_o}u_{Go}\right)-\frac{4}{\rho_L D_H\epsilon_o}\left(\tau_{Gw(\epsilon)}+\frac{\tau_{Gw(u_L)}}{1-\epsilon_o}u_{Lo}-\frac{\tau_{Gw(u_G)}}{\epsilon_o}u_{Go}\right)+\frac{4\tau_{Lwo}}{\rho_L D_H(1-\epsilon_o)^2}+\frac{4\tau_{Gwo}}{\rho_L D_H\epsilon_o^2}$$

$$K_3\triangleq\frac{1}{1-\epsilon_o}+\frac{\rho^*_G}{\epsilon_o}+\frac{C_{vm}}{\epsilon_o(1-\epsilon_o)^2}$$

$$K_4\triangleq 2\left[\left(\rho^*_G+\frac{C_{vm}}{1-\epsilon_o}\right)\frac{u_{Go}}{\epsilon_o}+\left(1+\frac{C_{vm}}{1-\epsilon_o}\right)\frac{u_{Lo}}{1-\epsilon_o}\right]+\frac{1}{\rho_L(1-\epsilon_o)}\left(-\Delta p_{Li(u_L)}+\sigma_{s(u_L)}-\tau^{Re}_{L(u_L)}+\rho_L\frac{C_{m1}}{1-\epsilon_o}u_{ro}\right)-\frac{1}{\rho_L\epsilon_o}\left(-\Delta p_{Li(u_g)}+\sigma_{s(u_g)}-\tau^{Re}_{L(u_g)}-\rho_L\frac{C_{m1}}{1-\epsilon_o}u_{ro}\right)$$

$$K_5\triangleq\left(\rho^*_G+\frac{C_{vm}}{1-\epsilon_o}\right)\frac{u^2_{Go}}{\epsilon_o}+\left(1+\frac{C_{vm}}{1-\epsilon_o}\right)\frac{u^2_{Lo}}{1-\epsilon_o}+\frac{\tau^{Re}_{Lio}}{\rho_L\epsilon_o(1-\epsilon_o)}+\frac{1}{\rho_L}\left(-\Delta p_{Li(\epsilon)}+\sigma_{s(\epsilon)}-\tau^{Re}_{L(\epsilon)}+\frac{(\Delta p_{Lio}+\tau^{Re}_{Lo})}{1-\epsilon_o}+\frac{\sigma_{so}}{\epsilon_o}+\frac{\rho_L C_{m2}}{\epsilon_o(1-\epsilon_o)}u^2_{ro}\right)+\frac{u_{Lo}}{\rho_L(1-\epsilon_o)}\left(-\Delta p_{Li(u_L)}+\sigma_{s(u_L)}-\tau^{Re}_{L(u_L)}+\frac{\rho_L C_{m1}}{1-\epsilon_o}u_{ro}\right)-\frac{u_{Go}}{\rho_L\epsilon_o}\left(-\Delta p_{Li(u_G)}+\sigma_{s(u_G)}-\tau^{Re}_{L(u_G)}-\frac{\rho_L C_{m1}}{1-\epsilon_o}u_{ro}\right).$$

Eq. (149) can be rewritten in a more compact form as:

$$\left[\frac{\partial}{\partial t}+a_{+}\frac{\partial}{\partial z}+T\left(\frac{\partial}{\partial t}+r_{-}\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t}+r_{+}\frac{\partial}{\partial z}\right)\right]\epsilon'=0, \tag{150}$$

where

$$a_{+}=\frac{K_2}{K_1} \tag{151}$$

$$r_{\pm}=-\frac{K_4}{2K_3}+\sqrt{\frac{1}{4}\left(\frac{K_4}{K_3}\right)^2-\left(\frac{K_5}{K_3}\right)} \tag{152}$$

$$T=\frac{K_3}{K_1}. \tag{153}$$

It can be shown (Whitham, 1974) that a_{+}, $r_{\pm}$ and T are the kinematic wave speed, the two void wave characteristics and the relaxation time of the void wave perturbation, respectively.

The behavior of the linear void wave may be conveniently examined by evaluating the dispersion relation. The dispersion relation can be obtained by assuming a modal traveling-wave solution of Eq. (150) of the form:

$$\epsilon'=\epsilon_{\mathrm{m}}\mathrm{e}^{j(\kappa z-\omega t)} \tag{154}$$

where κ and ω are the wave number and the angular frequency, respectively.

Inserting Eq. (154) into (150), we obtain the void wave's dispersion relation as,

$$j(\omega-a_{+}\kappa)+T(\omega-r_{-}\kappa)(\omega-r_{+}\kappa)=0. \tag{155}$$

If we consider the region where Eq. (150) is hyperbolic (i.e. where $r_{\pm}$ are real), the celerity, C_{ϵ}, of the void wave perturbation for traveling waves (i.e. when κ is real) can be found by solving the following coupled equations:

$$\omega_{\mathrm{I}}=\frac{1}{2T}\left(\frac{a_{+}-\bar{r}}{C_{\epsilon}-\bar{r}}-1\right) \tag{156}$$

$$\omega_{\mathrm{R}}^2=\frac{C_{\epsilon}^2}{4T^2}\left[\frac{(a_{+}-\bar{r})^2-(C_{\epsilon}-\bar{r})^2}{(C_{\epsilon}-\bar{r})^2(C_{\epsilon}-r_{-})(C_{\epsilon}-r_{+})}\right], \tag{157}$$

where

$$\omega=\omega_{\mathrm{R}}+j\omega_{\mathrm{I}}$$
$$C_{\epsilon}=\omega_{\mathrm{R}}/\kappa$$
$$\bar{r}=\frac{r_{+}+r_{-}}{2}.$$

One finds from Eq. (157) that void wave dispersion is pronounced for large values of the relaxation time (T), since the wave speed is strongly dependent on the angular frequency (ω_R) when the relaxation time is large.

It has been found (Park et al., 1990a) that Eqs. (156) and (157) have two solutions for C_ϵ for a specified value of angular frequency, ω_R. The faster void wave (C_ϵ^+) is the predominant one and it yields the classical kinematic void wave speed (a_+) in the limit of zero frequency, where the damping also vanishes. The other one is a complementary void wave, (C_ϵ^-), which is slower than C_ϵ^+ and has relatively large damping. A complementary void wave speed (a_-) can be found from Eq. (157).

$$\lim_{\omega_R \to 0} C_\epsilon^- = a_- = r_+ + r_- - a_+. \tag{158}$$

The well-known stability criteria (Whitham, 1974) for the C_ϵ^+ wave can be recovered by noting that the damping coefficient given by Eq. (156) will be negative if,

$$r_- < a_+ < r_+. \tag{159}$$

The kinematic void wave speed can be obtained using Eq. (151) and the constitutive relations given by Eqs. (118), (124) and (125):

$$A_+^* \triangleq \frac{a_+ - u_{Lo}}{u_{Go} - u_{Lo}} = 1 - n\epsilon_o, \tag{160}$$

where

$$n = \left[\frac{3\frac{C_{Do}}{R_b} - u_{ro}\frac{C_{D(u_L)}}{R_b} + (1-\epsilon_o)\frac{C_{D(\epsilon)}}{R_b} + \frac{32(1-\epsilon_o)\tau_{Lwo}}{3\rho_L u_{ro}^2 D_H}}{2\frac{C_{Do}}{R_b} - \epsilon_o u_{ro}\frac{C_{D(u_L)}}{R_b} + (1-\epsilon_o)u_{ro}\frac{C_{D(u_G)}}{R_b} + \frac{64\epsilon_o\tau_{Lwo}}{3\rho_L u_{ro}^2 D_H}} \right]. \tag{161}$$

If we use the interfacial drag coefficient given by Eqs. (58) and (59), we obtain,

$$n = 7/4 \tag{162}$$

for distorted bubbles (Harmathy, 1960), and,

$$n = \left[\frac{2.0 + 2.5\mu_m + (0.275 + 0.0625\mu_m)Re_b^{0.75} + \left[\frac{2(1+\epsilon_o)\tau_{Lwo}R_b}{4.5\rho_L u_{ro}^2 D_H}\right] Re_b}{1 + 0.1750 Re_b^{0.75} + \left[\frac{2\epsilon_o\tau_{Lwo}R_b}{2.25\rho_L u_{ro}^2} D_H\right] Re_b} \right] \tag{163}$$

for undistorted spherical bubbles (Ishii and Zuber, 1979).

The void wave characteristics can be found by inserting the appropriate constitutive relations into Eq. (152). As expected, the characteristics obtained by this dispersion analysis are the same as those obtained by the system's eigenvalue analysis in the previous section, Eq. (136).

The dispersion analysis results are more general than those from an eigenvalue analysis, since the dispersion analysis includes all possible frequency dependent propagations of different order. Their interaction is as indicated in Eq. (150); in particular, the first-order waves are the kinematic waves and the second order waves are the characteristics.

If we neglect the wall shear in Eq. (163), we find that $1.7 < n < 3.0$ for all possible values of the bubble Reynolds number, Re_b, for undistorted spherical particles. To bound the possibilities, the dimensionless kinematic wave speed for $n = 1.7$ and 3.0 is shown in Fig. 5 along with the corresponding characteristics. According to the stability criteria given in Eq. (159), the kinematic wave is stable for all possible values of Re_b until A^*_+ intersects λ^*_+.

The relaxation time can be found from Eq. (153) as:

$$T = \frac{R_b}{9u_{ro}} \frac{[\epsilon_o(1-\epsilon_o) + C_{vm} + \rho^*_G(1-\epsilon_o)^2]Re_b}{\left[1 + 0.1750Re_b^{0.75} + \left(\frac{2\epsilon_o \tau_{Lwo} R_b}{2.25\rho_L u_{ro}^2} D_H\right) Re_b\right]}. \tag{164}$$

As discussed earlier, the dispersion relation yields a complementary kinematic void wave. If we use Eq. (158), the dimensionless form of the complementary kinematic void wave speed is given by:

$$A^*_- = \frac{a_- - u_{Lo}}{u_{Go} - u_{Lo}} = \lambda^*_+ + \lambda^*_- - A^*_+, \tag{165}$$

where $A^*_\pm$ are given by Eq. (136). The nondimensional complementary kinematic speed given by Eq. (165) is also shown in Fig. 5 for the two limiting values of n.

The temporal damping of the complementary kinematic wave can be found from Eqs. (156), (158) and (164) as:

$$\omega_I = -\frac{9u_{ro}}{R_b} \frac{\left[1 + 0.1750Re_b^{0.75} + \left(\frac{2\epsilon_o \tau_{Lwo} R_b}{2.25\rho_L u_{ro}^2} D_H\right)\right] Re_b}{[\epsilon_o(1-\epsilon_o) + C_{vm} + \rho^*_G(1-\epsilon_o)^2]Re_b} \tag{166}$$

It was found (Park et al., 1994) that the damping of the complementary kinematic wave was large for most two-phase flows.

It should be noted that, to date, no experimental verification exists of the presence of the complementary void wave. However, since the intersection of the eigenvalues (i.e. where $\lambda^*_+ = \lambda^*_-$) signals the onset for ill-posedness, the complementary wave speed at that point is

$$A^*_- = 2r^*_+ - A^*_+. \qquad (167]$$

Thus, the onset of ill-posedness is, in principle, measurable if both A^*_+ and A^*_- can be measured.

The characteristics and the kinematic wave speed derived here have been compared against experimental data (Bourè, 1988; Kytömaa and Brennen, 1991). As shown in Fig. 6, the characteristics (with $C_{vm} = C_p = 0.5$, $C_r = 0.2$, $C_i = 0.3$, $C_{m1} = C_{m2} = 0.1$) and the kinematic wave speed of distorted bubble drag law agree well with the experimental data at low void fraction region (i.e. $0 < \epsilon_G < 0.15$). As the void fraction is increased, however, the data and the theory have large discrepancy since the assumptions used in this derivation may not be valid anymore.

7. Nonlinear void wave analysis

When the magnitude of the disturbances in two-phase flow is large, the linear void wave approximation is no longer appropriate for describing void wave propagations. It is known that many important nonlinear void wave phenomena occur in two-phase flow, for example, pool swell (Vea and Lahey, 1978). Surprisingly, nonlinear void waves have only been studied by a few authors (Park et al., 1990b; Haley et al., 1991; Lahey, 1991).

Let us consider a one-dimensional coordinate system moving at celerity, C_s. Then, we may obtain the relationship between the physical and the moving coordinate variables using the following transformations:

$$\zeta = z - C_s t. \tag{168}$$

Using Eq. (168), we can recast the one-dimensional system of equations, Eq. (132), into the moving coordinate system as,

$$\mathbf{A}\frac{\partial \Phi}{\partial t} + (\mathbf{B} - C_s\mathbf{A})\frac{\partial \Phi}{\partial \zeta} = \mathbf{c}. \tag{169}$$

If we consider only time-invariant, fully-developed solutions in ζ–t plane, we may neglect the transient term in Eq. (169) to obtain:

$$(\mathbf{B} - C_s\mathbf{A})\frac{\mathrm{d}\Phi}{\mathrm{d}\zeta} = \mathbf{c}. \tag{170}$$

Since we are interested in void fraction propagation, we obtain the void wave equation by taking the first vector component of

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\zeta} = (\mathbf{B} - C_s\mathbf{A})^{-1}\mathbf{c}, \tag{171}$$

which, as shown in Appendix, is given by,

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\zeta} = \frac{G(\epsilon, u_L, u_G, C_s)}{H(\epsilon, u_L, u_G, C_s)}, \tag{172}$$

where

$$H(\epsilon, u_L, u_G, C_s) = \frac{[\rho_G^*(1-\epsilon) + C_{vm}]}{(1-\epsilon)}(C_s - u_G)^2 + \frac{(1-\epsilon+C_{vm})}{(1-\epsilon)^2}(C_s - u_L)^2 + B_2 u_r\left(\frac{C_s - u_G}{\epsilon} + \frac{C_s - u_L}{1-\epsilon}\right) - B_1 \mid u_r \mid u_r \tag{173}$$

$$G(\epsilon, u_L, u_G, C_s) = \frac{1}{2(1-\epsilon)D_H}\left[\frac{3}{8}\frac{D_H}{R_b}C_D \mid u_r \mid u_r - f_w \mid u_L \mid u_L\right]$$

$$=(1-\rho_G^*)g\cos\theta, \tag{174}$$

and B_1 and B_2 are given by Eq. (132).

As can be seen in Appendix, the system's characteristic equation, that is, $\det(\mathbf{B} - C_s\mathbf{A}) = 0$, is equivalent to,

$$H(\epsilon, u_L, u_G, C_s) = 0. \tag{175}$$

If we solve Eq. (175) for the celerity, C_s, we obtain,

$$C_s^* \triangleq \frac{C_s - u_L}{u_G - u_L} = V^* \pm \sqrt{\upsilon^*/\tau^*}, \tag{176}$$

where V^*, υ^* and τ^* are identical to the expressions given by Eqs. (137)–(139), respectively. Therefore, we find that if we select the speed of the moving coordinate system the same as the system's characteristics, the void fraction gradient may become infinite in the specified frame of reference. Since an infinite void fraction gradient is possible only for a sharp discontinuity in the void fraction profile, one may obtain the shock solutions by selecting $C_s^* = \lambda^*$ (Haley et al., 1991).

Similarly, since the void fraction gradient vanishes in Eq. (172) when

$$G(\epsilon, u_L, u_G, C_s) = 0 \tag{177}$$

the roots of Eq. (177) can be recognized as the steady-states.

To identify the steady-states in terms of void fraction, we must consider phasic continuity. If we subtract the second from the first vector components of Eq. (170), we obtain,

$$\frac{d}{d\zeta}[\epsilon u_G + (1-\epsilon)u_L] = 0. \tag{178}$$

Integration of Eq. (178) yields the result that the total volume flux, j, is a conserved quantity in the ζ–ϵ plane. That is,

$$j \triangleq \epsilon u_G + (1-\epsilon)u_L = \text{constant}. \tag{179}$$

We may obtain another conserved quantity, K by integrating the second vector component of Eq. (170), which is the gas phase continuity, to obtain:

$$K \triangleq \epsilon C_s + (1-\epsilon)u_L = \text{constant}. \tag{180}$$

Using Eqs. (179) and (180), we can obtain the phasic velocities in terms of the void fraction:

$$u_L = \frac{K - \epsilon C_s}{1-\epsilon} \tag{181}$$

$$u_G = C_s + \frac{j-K}{\epsilon}. \tag{182}$$

Inserting Eqs. (181) and (182) into Eq. (174), we obtain,

$$G(\epsilon) = \frac{1}{2(1-\epsilon)D_{\mathrm{H}}}\left[\frac{3}{8}\frac{D_{\mathrm{H}}}{R_{\mathrm{b}}}C_{\mathrm{D}}(\epsilon)\right]\left(\frac{C_{\mathrm{s}}-K}{1-\epsilon}+\frac{j-K}{\epsilon}\right)^2 - f_{\mathrm{w}}\left(\frac{K-\epsilon C_{\mathrm{s}}}{1-\epsilon}\right)^2. \tag{183}$$

Unfortunately, an analytic expression of the zeroes of $G(\epsilon)$ is not possible. However, it has been found numerically that $G(\epsilon)$ has at most two zeroes within the range of $0 < \epsilon < 1$, for all possible values of j, K, C_{s} and interfacial drag, n. We denote those zeroes by ϵ_1 and ϵ_2.

Since j and K have been found to be conserved quantities, we have, from Eqs. (179) and (180),

$$\epsilon_1 u_{\mathrm{G1}} + (1-\epsilon_1)u_{\mathrm{L1}} = \epsilon_2 u_{\mathrm{G2}} - (1-\epsilon_2)u_{\mathrm{L1}} \tag{184}$$

$$\epsilon_1 C_{\mathrm{s}} + (1+\epsilon_1)u_{\mathrm{L1}} = \epsilon_2 C_{\mathrm{s}} + (1-\epsilon_2)u_{\mathrm{L1}}. \tag{185}$$

Solving Eq. (185) for C_{s}, we obtain:

$$C_{\mathrm{s}} = \frac{(1-\epsilon_1)u_{\mathrm{L1}} - (1-\epsilon_2)u_{\mathrm{L2}}}{\epsilon_2-\epsilon_1} = \frac{j_{\mathrm{L1}} - j_{\mathrm{L2}}}{\epsilon_2-\epsilon_1}. \tag{186}$$

It is interesting to note that the celerity given by Eq. (186) is the same as the continuity shock speed derived from drift–flux theory by Wallis (1969).

Also, if we neglect wall friction, the shock speed referenced to the liquid phase velocity can be found from Eqs. (177) and (183)–(185) as,

$$\frac{C_{\mathrm{s}} - u_{\mathrm{L1}}}{u_{\mathrm{G1}} - u_{\mathrm{L1}}} = \frac{1-\epsilon_2}{\epsilon_2-\epsilon_2}\left[\epsilon_2\sqrt{\frac{(1-\epsilon_2)C_{\mathrm{D}}(\epsilon_2)}{(1-\epsilon_1)C_{\mathrm{D}}(\epsilon_1)}} - \epsilon_1\right]. \tag{187}$$

Another form of the shock speed can be found by solving $G(\epsilon_1) = G(\epsilon_2) = 0$ with Eqs. (184) and (185), resulting in:

$$C_{\mathrm{s}} - j = \frac{\sqrt{\frac{16}{3}R_{\mathrm{b}}\cos\theta(1-\rho_{\mathrm{G}}^*)}}{(1-\epsilon_2)/\epsilon_2 - (1-\epsilon_1)/\epsilon_1}\left[\frac{1-\epsilon_1}{\epsilon_2}\sqrt{\frac{1-\epsilon_1}{C_{\mathrm{D}}(\epsilon_1)}} - \frac{1-\epsilon_2}{\epsilon_1}\sqrt{\frac{1-\epsilon_2}{C_{\mathrm{D}}(\epsilon_2)}}\right]. \tag{188}$$

Thus, Eq. (188) can be used to determine the nonlinear void wave speed referenced to the center of volume velocity (j) when a step change of void fraction is specified.

As a special case, the quasi-static linear kinematic wave speed can be found from Eq. (187) when the void fraction change is small enough. That is,

$$A_+^*(\epsilon) = \lim_{\epsilon_2\to\epsilon_1=\epsilon}\frac{C_{\mathrm{s}} - u_{\mathrm{L1}}}{u_{\mathrm{G1}} - u_{\mathrm{L1}}} = 1 - n\epsilon, \tag{189}$$

where

$$n = \frac{3C_{\mathrm{D}} + (1-\epsilon)C_{\mathrm{D}(\epsilon)}}{2C_{\mathrm{D}}}. \tag{190}$$

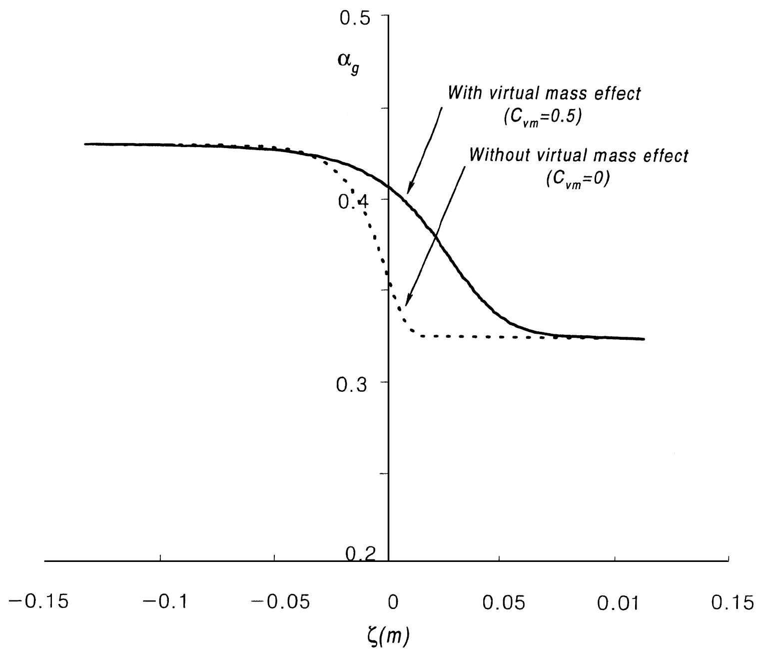

Fig. 7. Nonlinear void wave solutions with and without virtual mass.

This agrees with the result of the linear analysis, Eqs. (160) and (161), if we assume that the interfacial drag coefficient is only a function of void fraction [i.e. $C_{D(u_L)} = C_{D(u_G)} = 0$] and that wall friction is small.

It should be noted that when the interfacial drag law includes the phasic velocities (e.g. Eq. (59)), we need the expression for K to relate the phasic velocities with void fraction.

Since the total volume flux is assumed to be known, K can be obtained by solving $G(\epsilon_1) = G(\epsilon_2) = 0$ simultaneously, yielding:

$$K - j = \frac{\sqrt{\frac{18}{3} g R_b \cos\theta (1-\rho_G^*)}}{(1-\epsilon_2)/\epsilon_2 - (1-\epsilon_1)/\epsilon_1}\left[(1-\epsilon_1)\sqrt{\frac{1-\epsilon_1}{C_D(\epsilon_1)}} - (1-\epsilon_2)\sqrt{\frac{1-\epsilon_2}{C_D(\epsilon_2)}}\right]. \tag{191}$$

Consequently, we can obtain the phasic velocities in terms of void fractions using Eqs. (181), (182), (188) and (191).

If we eliminate the phasic velocities using conserved quantities, and Eqs. (181) and (182), we may obtain another form of Eq. (172), where void fraction is the only dependent variable:

$$\frac{d\epsilon}{d\zeta} = \frac{G(\epsilon)}{H(\epsilon)}. \tag{192}$$

Integrating Eq. (192) by separation of variables, we obtain an implicit expression for the void fraction profile in the ζ–ϵ plane as,

$$\zeta = \int_{\bar{\epsilon}}^{\epsilon} \frac{H(\epsilon')}{G(\epsilon')} d\epsilon', \tag{193}$$

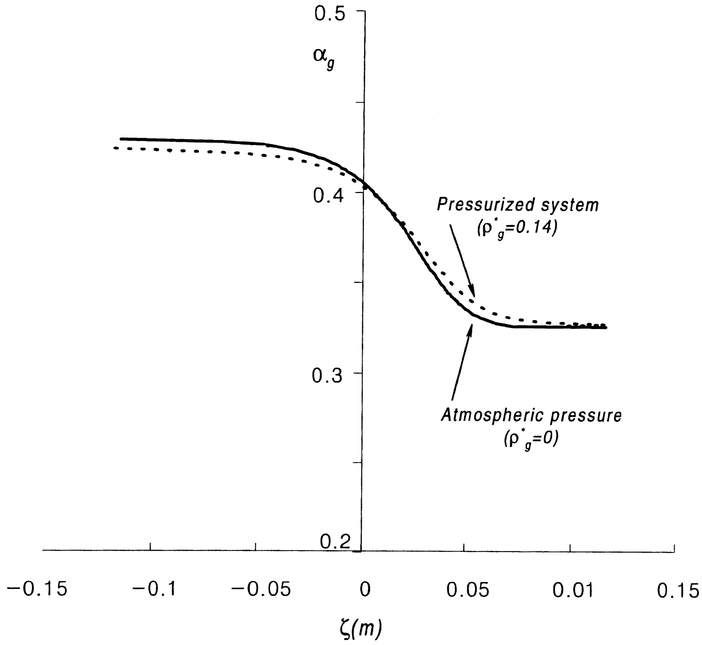

Fig. 8. Nonlinear void wave solutions at atmospheric and pressurized system (i.e. $p = 15.5$ MPa) conditions.

where

$$\bar{\epsilon} = \frac{(\epsilon_1 + \epsilon_2)}{2}$$

$$\epsilon_1 < \epsilon < \epsilon_2,$$

and ζ is taken so that $\zeta = 0$ at $\epsilon = \bar{\epsilon}$.

Thus, the void fraction profile determined by Eq. (193) propagates at celerity, C_s, given by Eq. (188) when the void fraction is changed from ϵ_1 to ϵ_2 (or, alternatively, from ϵ_2 to ϵ_1) without variation of the total volume flux, j.

Moreover, since we consider only time-invariant void waves, the resulting, solutions should be understood as being the fully-developed void wave profile for some specified initial and final conditions.

If we consider Eq. (193), we find that the void wave profile breaks (i.e. the solution is multivalued in the ζ–ϵ plane), when,

$$\epsilon_1 < \epsilon_{o1}, \epsilon_{o2}, \ldots < \epsilon_2 \tag{194}$$

where ϵ_{o1}, ϵ_{o2}, ... are the roots of $H(\epsilon) = 0$. More specifically, possible nonlinear void wave profiles, which depend on the integrand of Eq. (193). When $H(\epsilon)$ has zero between ϵ_1 and ϵ_2, the void wave profile breaks, which must be fitted to be a meaningful shock solution (Whitham, 1974). In contrast, when $H(\epsilon)$ has no zeroes between ϵ_1 and ϵ_2, the void wave profile is a soliton (i.e. a smooth but time invariant, propagating solution).

Theoretically (Whitham, 1974), wave breaking occurs when the propagating wave speed, C_s, is in between the larger characteristic speed of the initial state (i.e. conditions ahead of the shock, 1) and that of the final state (i.e. the conditions behind the shock, 2). That is,

$$(r_+)_1 = (\lambda_+^* u_r + u_L)_1 < C_s < (\lambda_+^* u_r + u_L)_2 = (r_+)_2 \tag{195}$$

when $\epsilon_1 > \epsilon_2$.

If we rearrange Eq. (195) using Eqs. (181), (182), (188) and (191), we obtain the condition for nonlinear void wave breaking as:

$$\frac{\lambda^*(\epsilon_2)}{1-\lambda_+^*(\epsilon_2)}(1-\epsilon_2)\epsilon_1 < \frac{\epsilon_2 - \epsilon_1}{1-\sqrt{\frac{C_D(\epsilon_1)}{C_D(\epsilon_2)}}\left(\frac{1-\epsilon_2}{1-\epsilon_1}\right)^{3/2}} < \frac{\lambda^*(\epsilon_1)}{1-\lambda_+^*(\epsilon_1)}(1-\epsilon_1)\epsilon_2. \tag{196}$$

Using the solutions of Eq. (193) with an undistorted bubble drag law ($n = 3.0$), void wave profile solutions with and without virtual mass are shown in Fig. 6. As can be seen, the virtual mass force reduces the void shock strength significantly. This observation is in agreement with that of Haley et al. (1991).

Nonlinear void wave solutions with two different values of the density ratio (ρ_G^*) are shown in Fig. 7. When $\rho_G^* = 0.14$, which is typical for the primary system pressure in PWR, is used, the void shock solitons do not change their shape significantly.

It is found that increased values of the interfacial stress make nonlinear wave solutions possible for a wider range of void fraction. However, the two-phase Reynolds stress does not change the properties of nonlinear void waves significantly.

The void fraction gradient parameter in the interfacial momentum exchange, C_{m2}, was found to be crucial in determining the behavior of nonlinear void waves.

The results of the nonlinear analysis imply that two-fluid model closure relations can be independently assessed and/or developed by investigating finite amplitude void waves. This is significant, since independent means are required for complete two-fluid model assessment.

8. Conclusion

An ensemble-averaged two-fluid model for adiabatic two-phase flows has been derived and used for the analysis of void wave propagation. A mechanistic treatment of the phasic interface has been found to be important for properly modeling interfacial momentum exchange phenomena. That is, the interfacial stress should be taken into account properly when the phasic momentum jump is considered.

Based upon this study, the continuous phase interfacial pressure difference and the void fraction gradient term in the non-drag force are found to be crucial in determining the behavior of non-linear two-phase bubbly flows. Linear and nonlinear void wave analysis reveals that void waves can be used as a means of assessing the closure relations for bubbly flows and flow regime transition. The authors hope this study will promote further research on two-fluid modeling as well as the investigation of void wave phenomena.

Appendix A

From the definitions of two-phase system matrices, Eq. (132), we obtain:

$$\mathbf{B} - C_s\mathbf{A} = \begin{pmatrix} C_{Gs} & \epsilon_G & 0 \\ C_{Ls} & 0 & 0 \\ B_L u_r^2 & \left(\rho_G^* + \frac{C_{vm}}{\epsilon_L}\right)C_{Gs} - B_2 u_r & -\left(1 + \frac{C_{vm}}{\epsilon_L}\right)C_{Ls} - B_2 u_r \end{pmatrix},$$

where

$$C_{Gs} \triangleq u_G - C_s$$
$$C_{Ls} \triangleq u_L - C_s.$$

The inverse of ($\mathbf{B} - \mathbf{C}_s\mathbf{A}$) is given by,

$$(\mathbf{B} - C_s\mathbf{A})^{-1} = \frac{1}{\Delta}\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix},$$

where

$$\Delta = \det(\mathbf{B} - C_s\mathbf{A}) = \epsilon_G\epsilon_L H(\epsilon, u_L, u_G, C_s) \quad \text{(A1)}$$

$$H(\epsilon, u_L, u_G, C_s) = \frac{\rho_G^*[(1-\epsilon) + C_{vm}]}{(1-\epsilon)}(C_s - u_G)^2 + \frac{1 - \epsilon + C_{vm}}{(1-\epsilon)^2}(C_s - u_L)^2 + B_2 u_r\left(\frac{C_s - u_G}{\epsilon} + \frac{C_s - u_L}{1-\epsilon}\right) - B_1 \mid u_r \mid u_r \quad \text{(A2)}$$

$$B_{11} = -\epsilon_{\rm L}\left(\rho_{\rm G}^* + \frac{C_{\rm vm}}{\epsilon_{\rm L}}\right)C_{\rm Gs} + B_2\epsilon_{\rm L}u_{\rm r}$$

$$B_{12} = \epsilon_{\rm G}\left(1 + \frac{C_{\rm vm}}{\epsilon_{\rm L}}\right)C_{\rm Ls} - B_2\epsilon_{\rm G}u_{\rm r}$$

$$B_{13} = -\epsilon_{\rm L}\epsilon_{\rm G}$$

$$B_{21} = \left(1 + \frac{C_{\rm vm}}{\epsilon_{\rm L}}\right)C_{\rm Ls}^2 - B_2u_{\rm r}C_{\rm Ls} - B_1\epsilon_{\rm L} \mid u_{\rm r} \mid u_{\rm r}$$

$$B_{22} = -\left(1 + \frac{C_{\rm vm}}{\epsilon_{\rm L}}\right)C_{\rm Gs}C_{\rm Ls} + B_2u_{\rm r}C_{\rm Gs}$$

$$B_{23} = \epsilon_{\rm L}C_{\rm Gs}$$

$$B_{31} = \left(\rho_{\rm G}^* + \frac{C_{\rm vm}}{\epsilon_{\rm L}}\right)C_{\rm Gs}C_{\rm Ls} - B_2u_{\rm r}C_{\rm Ls}$$

$$B_{32} = -\left(\rho_{\rm G}^* + \frac{C_{\rm vm}}{\epsilon_{\rm L}}\right)C_{\rm Gs}^2 + B_2u_{\rm r}C_{\rm Gs} + B_2\epsilon_{\rm G} \mid u_{\rm r} \mid u_{\rm r}$$

$$B_{33} = -\epsilon_{\rm G}C_{\rm Ls}.$$

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